K-closedness results for noncommutative L_p-spaces with filtrations

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In this paper, we establish new K-closedness results in the context of real interpolation of L_p-spaces associated with tracial von Neumann algebras equipped with filtrations. The main results adapt Bourgain's approach to the real interpolation of classical Hardy spaces on the disk within the framework of noncommutative martingales. As an application, we derive K-closedness results for various classes of martingale Hardy spaces, addressing a problem raised by Randrianantoanina.

Contents

1 Introduction

This paper is motivated by advances in the context of real interpolation theory of classical Hardy spaces on the disk following the work of Peter Jones. Let us review the results obtained in this context. Let $\mathbb{T}$ be the unit circle, and let $H_p(\mathbb{T})$ denote the associated Hardy space on the unit disk, viewed as a closed subspace of the Lebesgue space $L_p(\mathbb{T})$. Peter Jones established in [6] that there is a universal constant $C > 0$ such that for every $1 \leq p, q \leq \infty$, $f \in H_1(\mathbb{T}) + H_\infty(\mathbb{T})$, and $t > 0$, we have

$$K(t, f, H_p(\mathbb{T}), H_q(\mathbb{T})) \leq CK(t, f, L_p(\mathbb{T}), L_q(\mathbb{T})) \tag{1.1}$$

where K refers here to Peetre's K-functional in the context of real interpolation theory. According to the terminology introduced by Pisier in [9], one can reformulate Jones' theorem by saying that the subcouple $(H_p(\mathbb{T}), H_q(\mathbb{T}))$ is K-closed in the compatible couple $(L_p(\mathbb{T}), L_q(\mathbb{T}))$. When $1 < p, q < \infty$ then the estimate (1.1) is actually a direct consequence of the fact that the orthogonal projection of $L_2(\mathbb{T})$ onto $H_2(\mathbb{T})$, that is the Riesz projection, is L_p-bounded for $1 < p < \infty$. Thus, the essential contribution of Jone's theorem lies in the cases $p = 1$ or $q = \infty$, i.e. when the Riesz projection is no longer bounded. The estimate (1.1) contains the fact that for every $0 < \theta < 1$, we have

$$(H_1(\mathbb{T}), H_\infty(\mathbb{T}))_{\theta,p} = H_p(\mathbb{T}) \tag{1.2}$$

with equivalent norms, where $1/p = 1-\theta$, and where the notation on the left-hand side refers to the real interpolation method. In contrast with the existing proofs of Jones' result in the extensive literature devoted to the real interpolation of classical Hardy spaces on the disk so far, Bourgain was able to replace complex variable techniques with real variable methods. The approach of Bourgain to Jones' theorem is essentially based on the fact the Riesz projection is a Calderón-Zygmund singular integral operator. Using the Calderón-Zygmund decomposition, as well as the L_p-boundedness of the Riesz projection, he established that the subcouple $(H_1(\mathbb{T}), H_p(\mathbb{T}))$ is K-closed in the compatible couple $(L_1(\mathbb{T}), L_p(\mathbb{T}))$, for every $1 < p < \infty$. Then, using an abstract duality lemma for K-closedness due to Pisier in [9], he deduced that the subcouple $(H_p(\mathbb{T}), H_\infty(\mathbb{T}))$ is K-closed in $(L_p(\mathbb{T}), L_\infty(\mathbb{T}))$, for every $1 < p < \infty$. As highlighted by Kislyakov and Xu in [7], who established an abstract Wolff-type interpolation result for K-closedness, one can deduce Jones' theorem from the two partial results of Bourgain above.

The main contribution of this paper is to adapt Bourgain's approach in the setting of noncommutative martingales. To better explain our considerations, we now introduce the mathematical setting of the paper. We refer to the body of the paper for unexplained notations in the following. Let M be a tracial von Neumann algebra equipped with a filtration and let $(D_n)_{n\geq 1}$ denote the associated increment projections. For a fixed set of positive integers I, let $L_p^{\mathrm{sub}}(M)$ denote the closed subspace of $L_p(M)$ of elements $x \in L_p(M)$ such that $D_n(x) = 0$ for every $n \notin I$. Then we establish that the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in $(L_p(M), L_q(M))$ for every $1 \leq p, q \leq \infty$. We also establish an analogous result when M is equipped with two filtrations. The situation is totally analogous to the one with Hardy spaces on the disk, because the orthogonal projection of $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$, which is a particular instance of martingale transform, is L_p-bounded for $1 < p < \infty$, as established by Randrianantoanina in [11]. The decisive step in our arguments is to use the version of Gundy's decomposition theorem for martingales proved by Parcet and Randrianantoanina in [8], which provides the martingale counterpart of the classical Calderón-Zygmund decomposition.

The paper details two further contributions relative to square function inequalities for noncommutative martingales. We use the framework of column-row-mixed sequence spaces denoted $L_p(M, \ell_2)$ as introduced by Pisier and Xu in [10].

• Let $L_p^{\mathrm{ad}}(M, \ell_2)$ denote the closed subspace of $L_p(M, \ell_2)$ of adapted sequences. These spaces are closely related with Stein's inequality in the context of martingales inequalities. Indeed, one of the consequences of Stein's inequality is that the subcouple $(L_p^{\mathrm{ad}}(M, \ell_2), L_q^{\mathrm{ad}}(M, \ell_2))$ is complemented in $(L_p(M, \ell_2), L_q(M, \ell_2))$ for every $1 < p, q < \infty$. In particular, the subcouple $(L_p^{\mathrm{ad}}(M, \ell_2), L_q^{\mathrm{ad}}(M, \ell_2))$ is K-closed in the compatible couple $(L_p(M, \ell_2), L_q(M, \ell_2))$ for every $1 < p, q < \infty$. The first contribution of the paper is to extend this result for every $1 \leq p, q \leq \infty$. In the setting of row or column spaces, this result already appears in [12], but our approach allows us to encompass the case of mixed spaces as well.

• Let $L_p^{\mathrm{hardy}}(M, \ell_2)$ denote the closed subspace of $L_p(M, \ell_2)$ of martingale increment sequences. These spaces are connected with the usual martingale Hardy spaces in relation to the Burkholder-Gundy inequality. The second contribution of the paper is to establish that the subcouple $(L_p^{\mathrm{hardy}}(M, \ell_2), L_q^{\mathrm{hardy}}(M, \ell_2))$ is K-closed in the compatible couple $(L_p(M, \ell_2), L_q(M, \ell_2))$, for every $1 \leq p, q \leq \infty$. This answers a problem raised by Randrianantoanina in [12]. As a by-product of our result, for every $0 < \theta < 1$, we have

$$(L_1^{\mathrm{hardy}}(M, \ell_2), L_\infty^{\mathrm{hardy}}(M, \ell_2))_{\theta,p} = L_p^{\mathrm{hardy}}(M, \ell_2) \tag{1.3}$$

with equivalent norms, where $1/p = 1 - \theta$. In the setting of row or column spaces, the equality (1.3) has also recently been proved by Randrianantoanina in [13] using a different approach.

2 Preliminaries

In this first section, we recall some basic facts and classical results on interpolation theory, noncommutative L_p-spaces and noncommutative martingales. The aim is not to provide an exhaustive treatment, but rather to collect the notions and tools that will be used in the paper. The material of this section is mainly taken from [3], [5], [1].

2.1 Abstract interpolation theory

2.1.1 Compatible couples

A *compatible couple* is a couple (E_0, E_1) of subspaces of a common Hausdorff topological vector space E, such that E_j is equipped with a complete norm that makes the inclusion $E_j \to E$ continuous, for $j \in \{0, 1\}$. Then the intersection space $E_0 \cap E_1$ and the sum space E_0+E_1 are canonically equipped with the complete norms $\|\cdot\|_{E_0\cap E_1}$ and $\|\cdot\|_{E_0+E_1}$ defined as follows,

$$\|u\|_{E_0\cap E_1} := \max\Big\{\|u\|_{E_0},\ \|u\|_{E_1}\Big\}, \qquad \text{for } u \in E_0 \cap E_1.$$

$$\|u\|_{E_0+E_1} := \inf\Big\{\|u_0\|_{E_0}+\|u_1\|_{E_1} \ \big|\ u = u_0+u_1,\ u_0 \in E_0, u_1 \in E_1\Big\}, \qquad \text{for } u \in E_0+E_1.$$

An *intermediate space* for a compatible couple (E_0, E_1) is a subspace E_θ of E_0+E_1 that contains $E_0\cap E_1$, and that is equipped with a complete norm that makes the inclusions $E_0\cap E_1 \to E_\theta$ and $E_\theta \to E_0+E_1$ both continuous. If E_{θ_0}, E_{θ_1} are intermediate spaces for a compatible couple (E_0, E_1), then their sum $E_{\theta_0} + E_{\theta_1}$ and their intersection $E_{\theta_0} \cap E_{\theta_1}$ are also intermediate spaces for (E_0, E_1) when equipped with the corresponding sum norm $\|\cdot\|_{E_{\theta_0}+E_{\theta_0}}$ and intersection norm $\|\cdot\|_{E_{\theta_0}\cap E_{\theta_1}}$ as defined above.

2.1.2 Compatible bounded operators

Let (E_0, E_1) and (F_0, F_1) be two compatible couples. A *compatible bounded operator* $(E_0, E_1) \to (F_0, F_1)$ is an operator $T : E_0 + E_1 \to F_0 + F_1$ such that, if $j \in \{0, 1\}$, then T that maps E_j into F_j, and $T : E_j \to F_j$ is bounded. In this situation, we set

$$\|T\|_{(E_0,E_1)\to(F_0,F_1)} := \max\Big\{\|T\|_{E_0\to F_0}, \|T\|_{E_1\to F_1}\Big\}.$$

Let $T : (E_0, E_1) \to (F_0, F_1)$ be a compatible bounded operator. Note that T is injective/surjective/bijective if and only if $T : E_j \to F_j$ is, for $j \in \{0, 1\}$.

We say that T is an *embedding/quotient* of compatible couples if $T : E_j \to F_j$ is an embedding/quotient of normed spaces for $j \in \{0, 1\}$ (recall that a bounded operator

$T : E \to F$ between normed spaces is an embedding/quotient if it is injective/surjective and the induced bounded operator $E/\ker T \to \operatorname{ran} T$ is an isomorphism of normed spaces). We say that T is an *isomorphism* of compatible couples if $T : E_j \to F_j$ is an isomorphism of normed spaces, for $j \in \{0, 1\}$.

We say that T is *contractive* if $\|T\|_{(E_0,E_1)\to(E_0,E_1)} \leq 1$. We say that T is an *isometric embedding/coisometric quotient* of compatible couples if $T : E_j \to F_j$ is an isometric embedding/coisometric quotient of normed spaces for $j \in \{0, 1\}$ (recall that a quotient of normed spaces $T : E \to F$ is coisometric if the induced isomorphism of normed spaces $E/\ker T \to F$ is isometric). We say that T is an *isomorphism isomorphism* of compatible couples if $T : E_j \to F_j$ is an isometric isomorphism of normed spaces, for $j \in \{0, 1\}$.

Remark 2.1. There is an obvious way to define the category of compatible couples and compatible (contractive) bounded operators. The isomorphisms in this category correspond to the (isometric) isomorphisms of compatibles couples.

An *interpolation space* with constant $C \geq 1$ for a compatible couple (E_0, E_1) is an intermediate space E_θ for (E_0, E_1), such that, if $T : (E_0, E_1) \to (E_0, E_1)$ is a compatible bounded operator, then T maps E_θ into itself and the operator $T : E_\theta \to E_\theta$ is bounded, with $\|T\|_{E_\theta\to E_\theta} \leq C\|T\|_{(E_0,E_1)\to(E_0,E_1)}$. An *exact interpolation space* is an interpolation space with constant $C = 1$. The sum/intersection of (exact) interpolation spaces is again an (exact) interpolation space.

More generally, an *interpolation pair* with constant $C \geq 1$ for a pair of compatible couples (E_0, E_1) and (F_0, F_1) is a pair of intermediate spaces E_θ and F_θ for (E_0, E_1) and (F_0, F_1) respectively, such that, if $T : (E_0, E_1) \to (F_0, F_1)$ is a compatible bounded operator, then T maps E_θ into F_θ and the operator $T : E_\theta \to F_\theta$ is bounded, with $\|T\|_{E_\theta\to F_\theta} \leq C\|T\|_{(E_0,E_1)\to(F_0,F_1)}$. An *exact interpolation space* is an interpolation space with constant $C = 1$.

2.1.3 Interpolation functors

An *interpolation functor* with constant $C \geq 1$ is a map $\mathcal{F}$ that assigns to each compatible couple (E_0, E_1) an intermediate space $\mathcal{F}(E_0, E_1)$, such that, if (E_0, E_1), and (F_0, F_1) is a pair of compatible couples, then $\mathcal{F}(E_0, E_1)$ and $\mathcal{F}(F_0, F_1)$ is an exact interpolation pair with constant C for (E_0, E_1) and (F_0, F_1) (in this situation, if (E_0, E_1) is a compatible couple, then $\mathcal{F}(E_0, E_1)$ is necessarily an interpolation space with constant C for (E_0, E_1)). An exact interpolation functor is an interpolation functor with constant $C = 1$.

Remark 2.2. For instance, the map Σ (resp. Δ) that assings to each compatible couple

(E_0, E_1) the sum space $E_0 + E_1$ (resp. the intersection space $E_0 \cap E_1$) is an exact interpolation functor.

Remark 2.3. If $\mathcal{F}$ is an (exact) interpolation functor, then $\mathcal{F}$ defines in a obvious way a functor from the category of compatible couples and compatible (contractive) bounded operators to the category of complete normed spaces and (contractive) bounded operators.

Theorem 2.4 (Aronszajn-Gagliavro)**.** *If E_θ is a (exact) interpolation space for a compatible couple (E_0, E_1), then there is a (exact) interpolation functor $\mathcal{F}$ such that $E_\theta = \mathcal{F}(E_0, E_1)$ with (equal) equivalent norms.*

2.1.4 Subcouples

A *subcouple* of a compatible couple (E_0, E_1) is a couple (A_0, A_1) where A_j is a closed subspace of E_j for $j \in \{0, 1\}$. In this situation, the couple (A_0, A_1) inherits a canonical structure of compatible couple, so that the inclusion $A_0 + A_1 \to E_0 + E_1$ becomes an isometric embedding of compatible couples. Thus, if $\mathcal{F}$ is an (exact) interpolation functor, then $\mathcal{F}(A_0, A_1) \subset \mathcal{F}(E_0, E_1)$ continuously (contractively), but the inclusion $\mathcal{F}(E_0, E_1) \to \mathcal{F}(E_0, E_1)$ may not be an embedding of normed spaces.

2.1.5 Complementation

A subcouple (A_0, A_1) of a compatible couple (E_0, E_1) is *(1-)complemented* if there is an compatible (contractive) bounded operator $P : (E_0, E_1) \to (E_0, E_1)$ such that $P : E_j \to E_j$ is idempotent with range A_j, for $j \in \{0, 1\}$. In this situation, if $\mathcal{F}$ is an (exact) interpolation functor, then the inclusion $\mathcal{F}(A_0, A_1) \to \mathcal{F}(E_0, E_1)$ is an (isometric) embedding of normed spaces, and, moreover, we have $\mathcal{F}(A_0, A_1) = \mathcal{F}(E_0, E_1) \cap (A_0 + A_1)$.

2.1.6 Duality

Let (E_0, E_1) and (F_0, F_1) be two compatible couples such that the couples of normed spaces (E_0, F_0) and (E_1, F_1) are equipped with pairings (in our setting, a pairing on a couple of normed spaces (A, B) is a nondegenerate bounded bilinear form on $A \times B$), and assume that the two pairings are *compatible* in the sense that they agree on $(E_0 \cap E_1) \times (F_0 \cap F_1)$. In this situation, the couple of normed spaces $(E_0 \cap E_1, F_0 + F_1)$ and $(E_0 + E_1, F_0 \cap F_1)$ are in a obvious way canonically equipped with a pairing.

Let (E_0, E_1) be a compatible couple. If E_θ is an intermediate space for (E_0, E_1) such that $E_0 \cap E_1$ is dense in E_θ, then

$$E_\theta^* := \Big\{\phi \in (E_0 \cap E_1)^*, \sup_{u \in E_0 \cap E_1,\ \|u\|_{E_\theta} \leq 1} |\phi(u)| < \infty\Big\}$$

is a subspace of $(E_0 \cap E_1)^*$ and is equipped with the complete norm $\|\cdot\|_{E_\theta^*}$ given by the expression

$$\|\phi\|_{E_\theta^*} = \sup_{u \in E_0 \cap E_1,\ \|u\|_{E_\theta} \leq 1} |\phi(u)|, \qquad \text{for } \phi \in (E_0 \cap E_1)^*.$$

Moreover, it is clear that the inclusion $E_\theta^* \to (E_0 \cap E_1)^*$ is continuous. As a consequence, if the compatible couple (E_0, E_1) is *regular*, i.e. if $E_0 \cap E_1$ is dense in E_j for $j \in \{0, 1\}$, then the couple (E_0^*, E_1^*) inherits a canonical structure of compatible couple. In this situation, we have $E_0^* + E_1^* = (E_0 \cap E_1)^*$ with equal norms and $E_0^* \cap E_1^* = (E_0 + E_1)^*$ with equal norms. As a consequence, if E_θ is an intermediate space for (E_0, E_1) such that $E_0 \cap E_1$ is dense in E_θ, then E_θ^* is an intermediate space for (E_0^*, E_1^*).

Proposition 2.5. *Let (E_0, E_1) and (F_0, F_1) be two regular compatible couples. If $T : (E_0, E_1) \to (F_0, F_1)$ is a compatible bounded operator, then there is a unique compatible bounded operator $T^* : (F_0^*, F_1^*) \to (E_0^*, E_1^*)$ such that $T^* : F_0^* \to F_1^*$ and $T^* : F_1^* \to E_1^*$*

If E is interpolation space alors E^* interpolation space !

AJOUUUTERRRR ICI LE FAIT QUE LE DUAL DUN OPERATOR COMPATIBLE EST BIEN COMPATIBLE !!!

2.2 Real interpolation

2.2.1 K-functionals

Let (E_0, E_1) be a compatible couple. The *K-functional* of $u \in E_0 + E_1$ is defined for $t > 0$ as

$$K_t(u) = K_t(u, E_0, E_1) := \inf \Big\{\|u_0\|_{E_0} + t\|u_1\|_{E_1} \ \ | \ \ u_0 \in E_0,\ u_1 \in E_1,\ u = u_0 + u_1\Big\}.$$

For fixed $t > 0$, K_t is an equivalent norm on $E_0 + E_1$. If (E_0, E_1) and (F_0, F_1) are two compatible couples and $T : (E_0, E_1) \to (F_0, F_1)$ a compatible bounded operator, then

$$K_t(Tu, F_0, F_1) \leq \|T\|_{(E_0,E_1)\to(F_0,F_1)} K_t(u, E_0, E_1)$$

for every $u \in E_0+E_1$ and $t > 0$. In particular, if (A_0, A_1) is a subcouple of a compatible couple (E_0, E_1), then we have $K_t(u, E_0, E_1) \leq K_t(u, A_0, A_1)$ for every $u \in A_0 + A_1$ and $t > 0$.

A *K-method parameter* is a complete normed space $\Phi(t)$ of (equivalent class of) Lebesgue measurable functions with variable $t \in \mathbb{R}_+^*$ such that,

- ▷ if $f(t), g(t) \in \Phi(t)$ with $|g(t)| \leq |f(t)|$ then $\|g(t)\|_{\Phi(t)} \leq \|f(t)\|_{\Phi(t)}$,
- ▷ the function $1 \wedge t$ belongs to $\Phi(t)$.

If $\Phi(t)$ is a K-method parameter and (E_0, E_1) is a compatible couple, then

$$K_\Phi(E_0, E_1) := \Big\{u \in E_0 + E_1 \;\; | \;\; K_t(u, E_0, E_1) \in \Phi(t)\Big\}$$

is a subspace of $E_0 + E_1$ and is equipped with the complete norm $\| \cdot \|_{K_\Phi(E_0,E_1)}$ given by the expression

$$\|u\|_{K_\Phi(E_0,E_1)} := \|K_t(u, E_0, E_1)\|_{\Phi(t)}, \qquad \text{for } u \in K_\Phi(E_0, E_1).$$

This construction defines an exact interpolation functor K_Φ called the *K-method* with parameter Φ.

A subcouple (A_0, A_1) of a compatible couple (E_0, E_1) is *K-complemented* with constant $C \geq 1$ if for every $u \in A_0 + A_1$, whenever $u = u_0 + u_1$ with $u_0 \in E_0$, $u_1 \in E_1$, then $u = u_0' + u_1'$ with $u_0' \in A_0$, $u_1' \in A_1$ and $\|u_0'\|_{E_0} \leq C\|u_0\|_{E_0}$, $\|u_1'\|_{E_1} \leq C\|u_1\|_{E_1}$.

Proposition 2.6. *Let (A_0, A_1) be a subcouple of a compatible couple (E_0, E_1). If (A_0, A_1) is complemented in (E_0, E_1), then it is K-complemented in (E_0, E_1).*

A subcouple (A_0, A_1) of a compatible couple (E_0, E_1) is *K-closed* with constant $C \geq 1$ if $K_t(u, A_1, A_1) \leq CK_t(u, E_0, E_1)$ for every $u \in A_0 + A_1$ and $t > 0$.

Proposition 2.7. *Let (A_0, A_1) be a subcouple of a compatible couple (E_0, E_1). Then (A_0, A_1) is K-complemented in (E_0, E_1) if and only if it is K-closed in (E_0, E_1) and $A_j = (A_0 + A_1) \cap E_j$ for $j \in \{0, 1\}$.*

Proposition 2.8. *If (A_0, A_1) is a K-closed subcouple of a compatible couple (E_0, E_1), then for every K-method parameter Φ, the inclusion $K_\Phi(A_0, A_1) \to K_\Phi(E_0, E_1)$ is an embedding of normed spaces. Moreover, we have $K_\Phi(A_0, A_1) = (A_0+A_1)\cap K_\Phi(E_0, E_1)$.*

A complete proof of the following useful result can be found in [5, Theorem 6.1].

Theorem 2.9 (Pisier's duality lemma)**.** *If (A_0, A_1) is a K-closed subcouple of a regular compatible couple (E_0, E_1), then $(A_0^\perp, A_1^\perp)$ is K-complemented in (A_0^*, A_1^*).*

2.2.2 The real method

Let $0 < \theta < 1$ and $1 \leq p \leq \infty$. Let $\Phi_{\theta,p}(t)$ denote the space of Lebesgue-measurable functions f with variable $t \in \mathbb{R}_+^*$ such that

$$\|f(t)\|_{\Phi_{\theta,p}(t)} := \|t^{-\theta} f(t)\|_{L_p(dt/t)} < \infty$$

Then $\Phi_{\theta,p}(t)$ is a K-parameter space. If (E_0, E_1) be a compatible couple, the *real interpolation space* $(E_0, E_1)_{\theta,p}$ is the K-method interpolation space $\Phi_{\theta,p}(E_0, E_1)$. By convention, we set $(E_0, E_1)_{0,p} := E_0$ and $(E_0, E_1)_{1,p} := E_1$ for every $1 \leq p \leq \infty$.

Proposition 2.10. *Let (E_0, E_1) be a compatible couple. Then $E_0 \cap E_1$ is dense in $(E_0, E_1)_{\theta,p}$ for every $0 < \theta < 1$ and $1 \leq p < \infty$.*

Theorem 2.11 (Duality Theorem)**.** *Let (E_0, E_1) be a regular compatible couple. If $0 < \theta < 1$ and $1 \leq p < \infty$ then $(E_0, E_1)_{\theta,p}^* = (E_0^*, E_1^*)_{\theta,q}$ with equivalent norms, with constants depending on θ only, and where $1 < q \leq \infty$ is such that $1/p + 1/q = 1$.*

A subinterval of $[1, \infty]$ is said to be *nontrivial* if it is not emty, nor a singleton. A compatible family $(E_p)_{p \in I}$ indexed by a closed nontrivial subinterval of $[1, \infty]$ is a *real interpolation scale* if $(E_p, E_q)_{\theta,r} = E_r$ with equivalent norms for every $p, q, r \in I$ and $0 < \theta < 1$ such that $p \neq q$ and $1/r = (1 - \theta)/p + \theta/q$.

Proposition 2.12. *Let I be a nontrivial subinterval of $[1, \infty]$ and let $(A_p)_{p \in I}$ be a subfamily of a compatible family $(E_p)_{p \in I}$. We make the following three assumptions:*

1. *$(E_p)_{p \in I}$ is a real interpolation scale.*
2. *$(A_p)_{p \in I}$ is K-closed in $(E_p)_{p \in I}$, i.e. if $p, q \in I$ then the subcouple (A_p, A_q) is K-closed in (E_p, E_q).*
3. *$(A_p + A_q) \cap E_r = A_r$ for every $p, q, r \in I$ with $r \in [p, q]$.*

Then $(A_p)_{p \in I}$ is a real interpolation scale.

Proof. Let $p, q, r \in I$ and $0 < \theta < 1$ such that $p \neq q$ and $1/r = (1 - \theta)/p + \theta/q$. Then the we know that the inclusion $(A_p, A_q)_{\theta,r} \to (E_p, E_q)_{\theta,r} = E_r$ is an embedding, with range $(A_p + A_q) \cap (E_p, E_q)_{\theta,r} = (A_p + A_q) \cap E_r = A_r$. □

2.2.3 Reiteration and Wolff interpolation

The following formula is due to Holmstedt. A proof can be found in [4][Theorem 2.1].

Lemma 2.13 (Holmstedt's formula)**.** *Let* (E_0, E_1) *be a compatible couple. We set*

$$E_{\theta_0} := (E_0, E_1)_{\theta_0, p_0} \qquad \text{and} \qquad E_{\theta_1} := (E_0, E_1)_{\theta_1, p_1},$$

where $0 \le \theta_0 < \theta_1 \le 1$ *and* $1 \le p_0, p_1 \le \infty$*. Then for* $u \in E_{\theta_0} + E_{\theta_1}$ *and* $t > 0$*, we have*

$$\begin{aligned} K_t(u, E_{\theta_0}, E_{\theta_1}) \underset{p_0,p_1}{\sim} & \, 1_{\{\theta_0 \neq 0\}} \|s^{-\theta_0} K_s(u, E_0, E_1) 1_{(0, t^{1/\eta})}(s)\|_{L_{p_0}(ds/s)} \\ & + 1_{\{\theta_1 \neq 1\}} t \|s^{-\theta_1} K_s(u, E_0, E_1) 1_{(t^{1/\eta}, \infty)}(s)\|_{L_{p_1}(ds/s)} \end{aligned}$$

where $\eta := \theta_1 - \theta_0$*.*

From Holmstedt's formula, one deduces the reiteration theorem for the real method.

Theorem 2.14 (Reiteration theorem)**.** *Let* (E_0, E_1) *be a compatible couple. We set*

$$E_{\theta_0} := (E_0, E_1)_{\theta_0, p_0} \qquad \text{and} \qquad E_{\theta_1} := (E_0, E_1)_{\theta_1, p_1},$$

where $0 \le \theta_0 < \theta_1 \le 1$ *and* $1 \le p_0, p_1 \le \infty$*. Let* $0 < \lambda < 1$ *and* $1 \le p \le \infty$*. Then,*

$$(E_{\theta_0}, E_{\theta_1})_{\lambda, p} = (E_0, E_1)_{\theta_\lambda, p}$$

with equivalent norms, where $\theta_\lambda := (1 - \lambda)\theta_0 + \lambda\theta_1$*.*

The following result is a direct consequence of the reiteration theorem.

Corollary 2.15. *Let* $I = [p_0, q_0]$ *be a nontrivial closed subinterval of* $[1, \infty]$ *and let* $(E_p)_{p \in I_0}$ *be a compatible family such that* $(E_{p_0}, E_{q_0})_{\theta, r} = E_r$ *with equivalent norms for every* $0 < \theta < 1$ *and* $1 \le r \le \infty$ *such that* $1/r = (1 - \theta)/p_0 + \theta/q_0$*. Then* $(E_p)_{p \in I_0}$ *is a real interpolation scale.*

We also have a reiteration-type theorem for K-functionals. Again, it is a direct consequence of Holmstedt's formula.

Theorem 2.16 (K-reiteration)**.** *Let* (A_0, A_1) *be a* K*-closed subcouple of a compatible couple* (E_0, E_1) *with constant* $C \ge 1$*. We set*

$$A_{\theta_0} := (A_0, A_1)_{\theta_0, p_0} \qquad \text{and} \qquad A_{\theta_1} := (A_0, A_1)_{\theta_1, p_1},$$

$$E_{\theta_0} := (E_0, E_1)_{\theta_0, p_0} \qquad \text{and} \qquad E_{\theta_1} := (E_0, E_1)_{\theta_1, p_1},$$

where $0 \le \theta_0 < \theta_1 \le 1$ *and* $1 \le p_0, p_1 \le \infty$*. Then* $(A_{\theta_0}, A_{\theta_1})$ *is* K*-closed in* $(E_{\theta_0}, E_{\theta_1})$ *with a constant depending on* C, p_0, p_1 *only.*

The following result is a direct consequence of the above theorem.

Corollary 2.17. *Let $I_0 = [p_0, q_0]$ be a nontrivial closed subinterval of $[1,\infty]$ and let $(A_p)_{p\in I_0}$ be a subfamily of a compatible family $(E_p)_{p\in I_0}$. Assume that $(E_p)_{p\in I_0}$ is a real interpolation scale and that (A_{p_0}, A_{q_0}) is K-closed in (E_{p_0}, E_{q_0}). Then $(A_p)_{p\in I_0}$ is K-closed in $(E_p)_{p\in I_0}$.*

The following interpolation result is due to Wolff. The original proof of the following result can be found in [14][Theorem 1].

Theorem 2.18 (Wolff interpolation)**.** *Let $(E_0, E_{\theta_0}, E_{\theta_1}, E_1)$ be a compatible family. Assume that*

$$E_{\theta_0} := (E_0, E_{\theta_1})_{\eta_0,p_0} \quad \text{and} \quad E_{\theta_1} := (E_{\theta_0}, E_1)_{\eta_1,p_1},$$

with equivalent norms, where $0 < \eta_0, \eta_1 < 1$ and $1 \le p_0, p_1 \le \infty$. Then

$$E_{\theta_0} = (E_0, E_1)_{\theta_0,p_0} \quad \text{and} \quad E_{\theta_1} = (E_0, E_1)_{\theta_1,p_1}$$

with equivalent norms, where $\theta_0 := \frac{\eta_0\eta_1}{1-\eta_0+\eta_0\eta_1}$ and $\theta_1 := \frac{\eta_1}{1-\eta_0+\eta_0\eta_1}$ are determined by the relations $\theta_1 = (1-\eta_1)\theta_0 + \eta_1$ and $\theta_0 = \eta_0\theta_1$.

The following result is a direct consequence of Wolff interpolation theorem for the real method.

Corollary 2.19. *Let I be a closed nontrivial subinterval of $[1,\infty]$ and let $(E_p)_{p\in I}$ be a compatible family. Assume that there is a decomposition $I = I_1 \cup \ldots \cup I_n$ where $I_1, \ldots, I_n$ are nontrivial closed subintervals of $[1,\infty]$ such that the intersection $I_k \cap I_{k+1}$ is nontrivial for every $k \in \{1, \ldots, n-1\}$, and such that the compatible family $(E_p)_{p\in I_k}$ is a real interpolation scale for every $k \in \{1, \ldots, n-1\}$. Then $(E_p)_{p\in I}$ is a real interpolation scale.*

We also have a Wolff-type theorem for K-complementation due to Kislyakov and Xu. A proof can be found in [7][Theorem 2].

Theorem 2.20 (Wolff K-interpolation)**.** *Let $(A_0, A_{\theta_0}, A_{\theta_1}, A_1)$ be a subfamily of a compatible family $(E_0, E_{\theta_0}, E_{\theta_1}, E_1)$ such that (A_0, A_{θ_1}) is K-complemented in (E_0, E_{θ_1}) and (A_{θ_0}, A_1) is K-complemented in (E_{θ_0}, E_1). In addition, assume that*

$$E_{\theta_0} := (E_0, E_{\theta_1})_{\eta_0,p_0} \quad \text{and} \quad E_{\theta_1} := (E_{\theta_0}, E_1)_{\eta_1,p_1},$$

$$A_{\theta_0} := (A_0, A_{\theta_1})_{\eta_0,p_0} \quad \text{and} \quad A_{\theta_1} := (A_{\theta_0}, A_1)_{\eta_1,p_1}$$

with equivalent norms, where $0 < \eta_0, \eta_1 < 1$ and $1 \le p_0, p_1 \le \infty$. Then (A_0, A_1) is K-complemented in (E_0, E_1).

The following result is a direct consequence of the above theorem.

Corollary 2.21. *Let I be a nontrivial closed subinterval of $[1,\infty]$ and let $(A_p)_{p\in I}$ be a subfamily of a compatible family $(E_p)_{p\in I}$. We make the following four assumptions:*

1. *$(A_p)_{p\in I}$ and $(E_p)_{p\in I}$ are real interpolation scales.*
2. *There is a decomposition $I = I_1 \cup \ldots \cup I_n$ where $I_1, \ldots, I_n$ are nontrivial closed subintervals of $[1,\infty]$ such that the intersection $I_k \cap I_{k+1}$ is nontrivial for every $k \in \{1,\ldots,n-1\}$, and such that the compatible family $(A_p)_{p\in I_k}$ is K-complemented in $(E_p)_{p\in I_k}$ for every $k \in \{1,\ldots,n-1\}$.*

Then $(A_p)_{p\in I}$ is K-complemented in $(E_p)_{p\in I}$.

2.3 L_p-spaces

2.3.1 Generalities

Let M be a *tracial von Neumann algebra*, i.e. a von Neumann algebra equipped with a normal semifinite faithful (n.s.f.) trace τ. Let H denote the Hilbert space on which M acts. A closed and densely defined operator x on H with polar decomposition $x = u|x|$ and spectral decomposition $|x| = \int_0^{+\infty} s de_s$ is *affiliated* with M if $u \in M$ and $e_s \in M$ for all $s > 0$. The *distribution function* of x is the right-continuous decreasing function of the variable $s > 0$ denoted λ_x such that

$$\lambda_x(s) = \tau(1 - e_s), \quad \text{for } s > 0.$$

The *singular function* of x is the right-continuous decreasing function of the variable $s > 0$ denote μ_x such that

$$\mu_x(s) := \inf\Big\{t > 0 \ : \ \lambda_x(t) \le s\Big\}, \quad \text{for } s > 0.$$

A closed and densely defined operator x on H is *τ-measurable* if it is affiliated with M and if its distribution function (or its singular function) takes at leat one finite value. Any element of M is τ-measurable. The set $L_0(M)$ of τ-measurable operators then admits a canonical structure of complete Hausdorff topological $*$-algebra, so that the inclusion $M \to L_0(M)$ is a continuous $*$-morphism with dense range, and τ is canonically extended to the positive part of $L_0(M)$ so that

$$\tau(x) = \int_0^{+\infty} \lambda_x(s)ds = \int_0^{+\infty} \mu_x(s)ds, \quad \text{for } x \in L_0(M)_+.$$

For every $x \in L_0(M)$ and $1 \le p \le \infty$ we set

$$\|x\|_p := \begin{cases} \left(\int_0^{+\infty} \lambda_x(s)ps^{p-1}ds\right)^{1/p} = \left(\int_0^{+\infty} \mu_x(s)^p ds\right)^{1/p} & \text{if } p < \infty \\ \inf\{s > 0 \ | \ \lambda_x(s) = 0\} = \sup_{s>0} \mu_x(s) & \text{if } p = \infty \end{cases}.$$

Then, for $1 \leq p \leq \infty$, the *Lebesgue space*

$$L_p(M) := \Big\{x \in L_0(M) \;\mid\; \|x\|_p < \infty\Big\}$$

is a subspace of $L_0(M)$ and $\|\cdot\|_p$ is a complete norm on $L_0(M)$ that makes the inclusion $L_p(M) \to L_0(M)$ continuous. Moreover, we have $\|x\|_1 = \tau(x)$ for every $x \in L_0(M)_+$ and $\|x\|_\infty = \|x\|_{B(H)}$ so that $L_\infty(M) = M$ with equal norms. In particular, the family $(L_p(M))_{p\in[1,\infty]}$ inherits a canonical structure of compatible family. In the sequel, if $1 \leq p_0, p_1 \leq \infty$ then we use the notations $(L_{p_0} \cap L_{p_1})(M)$ and $(L_{p_0} + L_{p_1})(M)$ as a shorthand for $L_{p_0}(M) \cap L_{p_1}(M)$ and $L_{p_0}(M) + L_{p_1}(M)$ respectively.

Lemma 2.22. *Let $x \in L_0(M)$. Then $x \in (L_1 + L_\infty)(M)$ if and only if for every $t > 0$, we have*

$$\int_0^t \mu_x(s)ds < \infty$$

and in that case we have

$$K_t(x, L_1(M), L_\infty(M)) = \int_0^t \mu_x(s)ds, \qquad \textit{for } t > 0.$$

An immediate consequence of this formula we get the following result, showing in particular that the compatible family $(L_p(M))_{p\in[1,\infty]}$ is a real interpolation scale.

Theorem 2.23. *If $0 < \theta < 1$ then $(L_1(M), L_\infty(M))_{\theta,p} = L_p(M)$ with equivalent norms, with constants depending on p only, where $1/p = (1 - \theta)$.*

2.3.2 Köthe duality

In this paragraph M is a von Neumann algebra equipped with a (n.s.f.) trace τ. Then the trace τ extends to a positive and contractive linear form on $L_1(M)$ still denoted τ.

If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then the *Köthe dual*

$$E^\times(M) := \Big\{y \in L_0(M) \;:\; \forall x \in E(M),\; xy \in L_1(M)\Big\}$$

is a subspace of $L_0(M)$ and is equipped with the complete norm $\|\cdot\|_{E^\times(M)}$ given by the expression

$$\|x\|_{E^\times(M)} = \sup_{x\in E(M),\; \|x\|_{E(M)}\leq 1} |\tau(xy)|, \qquad \text{for } x \in E^\times(M).$$

Then $E^\times(M)$ is actually an exact interpolation space for $(L_1(M), L_\infty(M))$.

Proposition 2.24. *If* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *with* $1/p_0 + 1/q_0 = 1$ *and* $1/p_1 + 1/q_1 = 1$ *then* $(L_{p_0} + L_{p_1})^\times(M) = (L_{q_0} \cap L_{q_1})(M)$ *and* $(L_{p_0} \cap L_{p_1})^\times(M) = (L_{q_0} + L_{q_1})(M)$ *with equal norms.*

Remark 2.25. Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. The *Köthe bidual* $E^{\times\times}(M)$ is the Köthe dual of $E^\times(M)$. If $x \in E(M)$ then $x \in E^{\times\times}(M)$ and $\|x\|_{E^{\times\times}(M)} = \|x\|_{E(M)}$, but in general, the inclusion $E(M) \to E^{\times\times}(M)$ may not be surjective. It is surjective if and only if $E(M)$ satisfies *Fatous's lemma*, i.e. if every increasing bounded net $(x_\alpha)_\alpha$ of $E(M)_+$ admits a least upper bound with $\|\sup_\alpha x_\alpha\|_{E(M)} = \sup_\alpha \|x_\alpha\|_{E(M)}$. For example, if $1 \le p_0, p_1 \le \infty$ then $(L_{p_0} + L_{p_1})(M)$ and $(L_{p_0} \cap L_{p_1})(M)$ satisfy Fatou's lemma.

Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. Then the bilinear form $E(M) \times E^\times(M) \to \mathbb{C}$, $(x, y) \mapsto \tau(xy)$ defines a canonical duality between $E(M)$ and $E^\times(M)$, called the *Köthe duality* between $E(M)$ and $E^\times(M)$.

Proposition 2.26. *Let* $E(M)$ *be an exact interpolation space for* $(L_1(M), L_\infty(M))$. *Then* $(L_1 \cap L_\infty)(M)$ *is weakly dense in* $E(M)$ *with respect to Köthe duality.*

Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. If $E^*(M)$ denote the dual of $E(M)$, the Köthe duality between $E(M)$ and $E^\times$ induces a canonical isometric operator $E^\times(M) \to E^*(M)$, but in general it may not be surjective. It is surjective if and only if the norm of $E(M)$ is *order-continuous*, i.e. if for every decreasing net $(x_\alpha)_\alpha$ of $E(M)_+$ such that $\inf_\alpha x_\alpha = 0$ then $\inf_\alpha \|x_\alpha\|_{E(M)} = 0$. Thus, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm, then the weak topology of $E(M)$ w.r.t. Köthe duality actually coincides with its usual weak topology. For example, if $1 \le p_0, p_1 < \infty$ then $(L_{p_0} + L_{p_1})(M)$ and $(L_{p_0} \cap L_{p_1})(M)$ have order-continuous norm.

2.4 Martingales

2.4.1 Conditional expectations

Let M be a tracial von Neumann algebra and let N be a von Neumann subalgebra of M such that there is a (trace-preserving normal faithful) conditional expectation E of M onto N. Then N becomes a tracial von Neumann algebra with the restricted trace such that $L_1(N)$ is a subspace of $L_1(M)$ and the inclusion operator $L_1(N) \to L_1(M)$ is isometric. Moreover, the conditional expectation E extends to a contractive compatible operator $(L_1(M), L_\infty(M)) \to (L_1(N), L_\infty(N))$ that restricts to the identity on $L_1(N) + L_\infty(N)$. As a consequence, if $\mathcal{F}$ is an exact interpolation functor then $\mathcal{F}(L_1(N), L_\infty(N))$ is a subspace of $\mathcal{F}(L_1(M), L_\infty(M))$ and the inclusion operator $\mathcal{F}(L_1(N), L_\infty(N)) \to \mathcal{F}(L_1(M), L_\infty(M))$ is isometric. As a consequence, if $\mathcal{F}$ is an exact interpolation functor, then the conditional expectation E induces a canonical contractive operator $\mathcal{F}(L_1(M), L_\infty(N)) \to \mathcal{F}(L_1(N), L_\infty(N))$ which restricts to the identity on $\mathcal{F}(L_1(N), L_\infty(N))$.

2.4.2 Filtrations and martingales

Let M be a tracial von Neumann algebra equipped with a *filtration*, i.e. an increasing sequence $(M_n)_{n\geq 1}$ of von Neumann subalgebras of M whose union $\cup_{n\geq 1} M_n$ is weak*-dense in M and such that there is a trace-preserving normal faithful conditional expectation E_n of M onto M_n for every $n \geq 1$. Then $(E_n)_{n\geq 1}$ is an increasing sequence of commuting projections. For every $n \geq 1$, we set

$$D_n := D_n - D_{n-1}$$

(with the convention $E_0 := 0$). Then $(D_n)_{n\geq 1}$ is a sequence of mutually orthogonal projections that commute with the $(E_n)_{n\geq 1}$. We will refer to them as the *increment projections* associated with the filtration.

A sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is *adapted* if $E_n(x_n) = x_n$ for all $n \geq 1$. A sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is a *martingale* if it is adapted and $E_{n-1}(x_n) = x_{n-1}$ for all $n \geq 2$, and in that case $E_k(x_n) = x_{k\wedge n}$ for every $n, k \geq 1$.

A sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is a *martingale increment* if it is adapted and $E_{n-1}(x_n) = 0$ for all $n \geq 2$, and in that case $E_k(x_n) = 1_{k\geq n} x_n$ for every $n, k \geq 1$.

If $x \in (L_1 + L_\infty)(M)$, the sequence $(E_n(x))_{n\geq 1}$ is a martingale, and the sequence $(D_n(x))_{n\geq 1}$ is a martingale increment. Note that we have $x \in \cup_{n\geq 1}(L_1 + L_\infty)(M_n)$ if and only if the sequence $(E_n(x))_{n\geq 1}$ is eventually constant, and also if and only if the sequence $(D_n(x))_{n\geq 1}$ is eventually zero.

Lemma 2.27. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. Then the subspace $\bigcup_{n\geq 1}(L_1 \cap L_\infty)(M_n)$ is weakly dense in $E(M)$ with respect to Köthe duality.*

Proof. $\bigcup_{n\geq 1}(L_1 \cap L_\infty)(M_n)$ is a $*$-subalgebra of $L_\infty(M)$. Moreover, it is clearly weak*-dense in $L_\infty(M)$ because $\cup_{n\geq 1} L_\infty(M_n)$ is, by definition. Thus $\bigcup_{n\geq 1}(L_1 \cap L_\infty)(M_n)$ is norm-dense in $L_1(M)$. As a consequence, it is weakly dense in $(L_1 \cap L_\infty)(M)$ and thus also in $E(M)$. $\square$

Theorem 2.28. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm.*

1. *If $x \in E(M)$, then the sequence $(E_n(x))_{n\geq 1}$ converges (in norm) to x in $E(M)$.*
2. *If $y \in E^{\times}(M)$, then the sequence $(E_n(y))_{n\geq 1}$ converges weakly to y in $E^{\times}(M)$ with respect to Köthe duality.*

Proof. Let $x \in E(M)$ and $\epsilon > 0$. By the previous lemma, we know that the subspace $\cup_{n\geq 1}(L_1 \cap L_\infty)(M_n)$ is weakly dense in $E(M)$, and thus it is norm-dense in $E(M)$ because $E(M)$ has order continuous norm. Thus, there is $y \in E(M)$ and $k \geq 1$ such that $\|x-y\|_{E(M)} < \epsilon$ and $E_k(y) = y$. Then, for all $n \geq k$, we have

$$\begin{aligned}\|E_n(x) - x\|_{E(M)} &= \|E_n(x) + E_n(y) + y - x\|_{E(M)} \\ &\leq \|E_n(x-y)\|_{E(M)} + \|x-y\|_{E(M)} \\ &\leq 2\|x-y\|_{E(M)} < 2\epsilon\end{aligned}$$

which shows that $(E_n(x))_{n\geq 1}$ converges in norm to x. Now, if $y \in E^{\times}(M)$ then for every $x \in E(M)$ we get

$$\tau(xE_n(y)) = \tau(E_n(x)y) \underset{n\to\infty}{\to} \tau(xy)$$

as desired. □

Corollary 2.29. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. If $x \in E(M)$ and $y \in E^{\times}(M)$, then*

$$\tau(xy) = \sum_{n=1}^{+\infty} \tau(D_n(x)D_n(y)).$$

2.4.3 Gundy's decomposition

Let M be a von Neumann algebra equipped with a (n.s.f.) trace τ and a filtration $(M_n)_{n\geq 1}$. Let $(E_n)_{n\geq 1}$ and $(D_n)_{n\geq 1}$ denote the associated conditional expectations and increment projections.

We state the following version of Gundy's decomposition theorem for martingales, adapted from [8][Corollary 2.10]. It will be an important tool for the main results of the paper.

Theorem 2.30 (Gundy's decomposition)**.** *Let $y \in (L_1 \cap L_2)(M)$ and $\lambda > 0$. Then there is a decomposition $y = a + b + c$ with $a, b, c \in L_2(M)$ and a adapted sequence $(p_n)_{n\geq 1}$ of projections of M such that*

- $\|a\|_2^2 \leq C^2 \lambda \|y\|_1$,
- $\sum_{n\geq 1} \|D_n(b)\|_1 \leq C\|y\|_1$,

- $\tau(1-p) \leq C\lambda^{-1}\|y\|_1$ *where* $p := \wedge_{n\geq 1} p_n$,
- $p_{n-1}D_n(c)p_{n-1} = 0$ *for every* $n \geq 1$ *(with the convention* $p_0 = 1$*), and in particular* $pD_n(c)p = 0$ *for every* $n \geq 1$.

where $C > 0$ *is a universal constant.*

For the convenience of the reader we provide a proof of this theorem. We will need the following well-known result which is fundamental in all the developments of martingale theory in the context of tracial von Neumann algebras. The proof of the estimate (2.2) is contained in [8][Proposition 1.5]. The proof of the estimate (2.3) is contained in [11][Lemma 3.4].

Lemma 2.31 (Cuculescu)**.** *Let* $y \in L_1(M)_+$ *and* $\lambda > 0$. *Then there is a decreasing sequence* $(p_n)_{n\geq 1}$ *of projections of* M *such that*

(i) for all $n \geq 1$, $p_n \in M_n$,

(ii) for all $n \geq 1$, $p_n E_n(y) p_n \leq \lambda p_n$.

The projections $(p_n)_{n\geq 1}$ *are refered as* ***Cuculescu's projections*** *associated with* x *and* λ. *Moreover, we have the following estimates*

$$\tau(1-p) \leq \lambda^{-1}\|y\|_1, \tag{2.1}$$

where $p := \wedge_{n\geq 1} p_n$,

$$\sum_{k=1}^{n} \|p_k D_k(y) p_k - p_{k-1} D_k(y) p_{k-1}\|_1 \leq 3\|y\|_1 \quad \textit{for } n \geq 1, \tag{2.2}$$

$$\sum_{k=1}^{n} \|p_k E_k(y) p_k - p_{k-1} E_{k-1}(y) p_{k-1}\|_2^2 \leq 3\lambda\|y\|_1 \quad \textit{for } n \geq 1, \tag{2.3}$$

with the convention $p_0 := 1$ *and* $E_0 := 0$.

Proof of Gundy's decomposition theorem. Let $y \in (L_1 \cap L_2)(M)$. First we assume that y is positive. Let $(p_n)_{n\geq 1}$ be the sequence of Cuculescu's projections associated with y and λ. By (2.1) we already know that $\tau(1-p) \leq \lambda^{-1}\|y\|_1$ where $p := \wedge_{n\geq 1} p_n$. For $n \geq 1$, we set

$$\begin{cases} da_n &= p_n D_n(y) p_n - E_{n-1}(p_n D_n(y) p_n) \\ db_n &= p_{n-1} D_n(y) p_{n-1} - da_n \\ dc_n &= D_n(y) - p_{n-1} D_n(y) p_{n-1} \end{cases}$$

with the convention $p_0 := 1$ and $E_0 := 0$. Then the sequences $(da_n)_{n\geq 1}$, $(db_n)_{n\geq 1}$, $(dc_n)_{n\geq 1}$ clearly are martingale increments, and it is clear that $D_n(y) = da_n + db_n + dc_n$

for every $n \geq 1$. Now, for $n \geq 1$, by (2.3) we have

$$\begin{aligned}
\Big\| \sum_{k=1}^{n} da_k \Big\|_2^2 &= \sum_{k=1}^{n} \|da_k\|_2^2 \\
&= \sum_{k=1}^{n} \|p_k D_k(y) p_k - E_{k-1}(p_k D_k(y) p_k)\|_2^2 \\
&\leq 4 \sum_{k=1}^{n} \|p_k D_k(y) p_k\|_2^2 \\
&= 4 \sum_{k=1}^{n} \|p_k E_k(y) p_k - p_k E_{k-1}(y) p_k\|_2^2 \\
&= 4 \sum_{k=1}^{n} \|p_k (p_k E_k(y) p_k - p_{k-1} E_{k-1}(y) p_{k-1}) p_k\|_2^2 \\
&\leq 4 \sum_{k=1}^{n} \|p_k E_k(y) p_k - p_{k-1} E_{k-1}(y) p_{k-1}\|_2^2 \\
&\leq 12\lambda \|y\|_1.
\end{aligned}$$

Thus the serie $\sum_{n\geq 1} da_n$ converges inconditionaly in $L_2(M)$, and if we denote $a \in L_2(M)$ its sum, we have $\|a\|_2^2 \leq 12\lambda\|y\|_1$. For $n \geq 1$, by (2.2) we also have

$$\begin{aligned}
\sum_{k=1}^{n} \|db_k\|_1 &\leq \sum_{k=1}^{n} \|p_k D_k(y) p_k - p_{k-1} D_k(y) p_{k-1}\|_1 + \sum_{k=1}^{n} \|E_{k-1}(p_k D_k(y) p_k)\|_1 \\
&= \sum_{k=1}^{n} \|p_k D_k(y) p_k - p_{k-1} D_k(y) p_{k-1}\|_1 \\
&+ \sum_{k=1}^{n} \|E_{k-1}(p_k D_k(y) p_k - p_{k-1} D_k(y) p_{k-1})\|_1 \\
&\leq 2 \sum_{k=1}^{n} \|p_k D_k(y) p_k - p_{k-1} D_k(y) p_{k-1}\|_1 \\
&\leq 6\|y\|_1.
\end{aligned}$$

Thus the serie $\sum_{n\geq 1} db_n$ converges absolutely in $L_1(M)$, and if we denote $b \in L_1(M)$ its sum, we have $\sum_{n\geq 1} \|D_n(b)\|_1 = \sum_{n\geq 1} \|db_n\|_1 \leq 7\|y\|_1$. We finally deduce that the serie $\sum_{n\geq 1} dc_n$ converges in $(L_1 + L_2)(M)$, and if we denote $c \in (L_1 + L_2)(M)$ its sum, we clearly have $x = a + b + c$ and

$$p_{n-1} D_n(c) p_{n-1} = p_{n-1} dc_n p_{n-1} = p_{n-1}(D_n(y) - p_{n-1} D_n(y) p_{n-1}) p_{n-1} = 0$$

for every $n \geq 1$, as desired. It remains to justify that $b, c \in L_2(M)$. As $y = a+b+c$, it suffices to justify that $b \in L_2(M)$. As $y \in L_2(M)$, we know that the serie $\sum_{n\geq 1} D_n(y)$

converges to y in $L_2(M)$, and thus the serie $\sum_{n\geq 1} p_{n-1}D_n(y)p_{n-1}$ also converges in $L_2(M)$ because $(p_{n-1}D_n(y)p_{n-1})_{n\geq 1}$ is a martingale increment and

$$\sum_{n\geq 1} \|p_{n-1}D_n(y)p_{n-1}\|_2^2 \leq \sum_{n\geq 1} \|D_n(y)\|_2^2 = \|y\|_2^2.$$

As we have proved that the serie $\sum_{n\geq 1} da_n$ converges to a in $L_2(M)$, it follows that the serie $\sum_{n\geq 1} db_n = \sum_{n\geq 1} p_{n-1}D_n(y)p_{n-1} - da_n$ converges in $L_2(M)$. As a consequence we have $b \in L_2(M)$ as desired. Now we drop the assumption of positivity. It is well-known that we can decompose $y = y_1 + y_2 + y_3 + y_4$ with $y_j \in (L_1 \cap L_2)(M)_+$, $\|y_j\|_1 \leq \|y\|_1$. Apply the preceding construction to y_j yields a sequence of projections $(p_n^j)_{n\geq 1}$ and a decomposition $y_j = a_j + b_j + c_j$ with $a_j, b_j, c_j \in L_2(M)$. Then it suffices to set $p_n := \wedge_j p_n^j$, $a := \sum_j a_j$, $b := \sum_j b_j$, $c := \sum_j c_j$ and to check that the required conditions are satisfied. $\square$

3 Statements of the main results

3.1 Theorem A

Let M be a tracial von Neumann algebra equipped with a filtration $(M_n)_{n\geq 1}$ with associated conditional expectations denoted $(E_n)_{n\geq 1}$ and associated increment projections denoted $(D_n)_{n\geq 1}$. Let I be a fixed set of positive integer. If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, we set

$$E^{\mathrm{sub}}(M) := \Big\{x \in E(M) \ : \ \forall n \notin I,\ D_n(x) = 0\Big\}.$$

It is clear that $E^{\mathrm{sub}}(M)$ is a weakly closed subspace of $E(M)$ w.r.t. Köthe duality, and in addition it is stabilised by E_n for every $n \geq 1$.

The first main result of the paper reads as follows.

Theorem A. *If $1 \leq p, q \leq \infty$, then the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-complemented in the compatible couple $(L_p(M), L_q(M))$ with a universal constant.*

In order to derive interesting consequences from Theorem A, we need some further results.

Proposition 3.1. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ such that either $E(M)$ has order continuous norm or $E(M) = F^\times(M)$ where $F(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then*

$$\Big\{x \in \cup_{n\geq 1}(L_1 \cap L_\infty)(M_n) \ : \ \forall n \notin I,\ D_n(x) = 0\Big\}$$

is a weakly dense subspace of $E^{\mathrm{sub}}(M)$ with respect to Köthe duality.

Proof. Fix $x \in E^{\mathrm{sub}}(M)$. Then we know that the sequence $(E_n(x))_{n\geq 1}$ belongs to $E^{\mathrm{sub}}(M)$, and by By Theorem 2.28 it converges weakly to x in $E(M)$ w.r.t. Köthe duality. Thus we can assume that there is $n \geq 1$ such that $E_n(x) = x$, so that we have

$$x = \sum_{k=1}^{n} D_k(x) = \sum_{k\in I, k\leq n} D_k(x).$$

As $(L_1 \cap L_\infty)(M)$ is weakly dense in $E(M)$, there is a net $(y_\alpha)_\alpha$ of $(L_1 \cap L_\infty)(M)$ that converges weakly to x in $E(M)$. We set

$$x_\alpha := \sum_{k\in I, k\leq n} D_k(y_\alpha).$$

Then $x_\alpha \in (L_1 \cap L_\infty)(M_n)$ and $D_n(x_\alpha) = 0$ for every $n \notin I$. As the net $(x_\alpha)_\alpha$ converges weakly to x in $E(M)$, the proof is complete. $\square$

Theorem 3.2. *Let $1 \leq p, q \leq \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(M) := K_\Phi(L_p(M), L_q(M))$ has order continuous norm. Then*

$$E^{\mathrm{sub}}(M) = K_\Phi(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$$

with equivalent norms, with universal constants.

Proof. As $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in $(L_p(M), L_q(M))$ with a universal constant, we know that the inclusion operator

$$K_\Phi(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M)) \to K_\Phi(L_p(M), L_q(M)) = E(M)$$

is an embedding of normed spaces, with universal constants, and with range $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$. Thus it suffices to show that $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$ is a norm-dense subspace of $E^{\mathrm{sub}}(M)$. First of all, it is clear that $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$ is indeed a subspace of $E^{\mathrm{sub}}(M)$. Besides, it clearly contains

$$\Big\{x \in \cup_{n \geq 1}(L_1 \cap L_\infty)(M_n) \ : \ \forall n \notin I, \ D_n(x) = 0\Big\}.$$

By the previous proposition, we deduce that $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$ is weakly dense in $E^{\mathrm{sub}}(M)$, and thus it is norm-dense because $E(M)$ has order continuous norm. The proof is complete. □

By considering in particular the real interpolation functors, we deduce the following result.

Corollary 3.3. *The compatible family $(L_p^{\mathrm{sub}}(M))_{p \in [1,\infty]}$ is a real interpolation scale.*

3.2 Theorem B

Let M be a tracial von Neumann algebra equipped with two filtrations $(M_n^-)_{n \geq 1}$, $(M_n^+)_{n \geq 1}$ with associated conditional expectations denoted by $(E_n^-)_{n \geq 1}$, $(E_n^+)_{n \geq 1}$ and increment projections denoted by $(D_n^-)_{n \geq 1}$, $(D_n^+)_{n \geq 1}$ respectively. Let I^-, I^+ be two fixed sets of positive integers. We add the two following assumptions.

Commutation Assumption. The filtrations $(M_n^-)_{n \geq 1}$, $(M_n^+)_{n \geq 1}$ *commute* in the sense that for every $m, n \geq 1$, we have

$$E_m^- E_n^+ = E_n^+ E_m^-. \tag{3.1}$$

Orthogonality Assumption. For every $m \notin I^-$ and $n \notin I^+$, we have

$$D_m^- D_n^+ = D_n^+ D_m^- = 0. \tag{3.2}$$

If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, as in the previous paragraph we set

$$E_{\pm}^{\mathrm{sub}}(M) := \Big\{x \in E(M) \ : \ \forall n \notin I^{\pm},\ D_n^{\pm}(x) = 0\Big\},$$

and finally we set

$$\begin{aligned} E^{\mathrm{sub}}(M) &:= E_{-}^{\mathrm{sub}}(M) \cap E_{+}^{\mathrm{sub}}(M) \\ &= \Big\{x \in E(M) \ : \ \forall n \notin I^{-},\ D_n^{-}(x) = 0,\ \forall n \notin I^{+},\ D_n^{+}(x) = 0\Big\}. \end{aligned}$$

It is clear that $E^{\mathrm{sub}}(M)$ is a weakly closed subspace of $E(M)$ w.r.t. Köthe duality, and in addition it is stabilised by $E_n^{\pm}$ for every $n \geq 1$.

The second main result of the paper reads as follows.

Theorem B. *If $1 < p, q \leq \infty$, then the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-complemented in the compatible couple $(L_p(M), L_q(M))$ with a constant depending on p, q only.*

The main difference with A is the restriction on the interval $(1, \infty]$ instead of the full interval $[1, \infty]$. As in the previous section, we can derive some consequences from Theorem B.

Lemma 3.4. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm.*

1. *If $x \in E(M)$ then $(E_n^- E_n^+(x))_{n \geq 1}$ converges in norm to x in $E(M)$ w.r.t. Köthe duality.*
2. *If $y \in E^{\times}(M)$ then $(E_n^- E_n^+(y))_{n \geq 1}$ converges *-weakly to y in $E^{\times}(M)$ w.r.t. Köthe duality.*

Proof. If $x \in E(M)$, then

$$\begin{aligned} \|E_n^- E_n^+(x) - x\|_{E(M)} &= \|E_n^-(E_n^+(x) - x) + E_n^-(x) - x\|_{E(M)} \\ &\leq \|E_n^+(x) - x\|_{E(M)} + \|E_n^-(x) - x\|_{E(M)} \underset{n \to \infty}{\to} 0. \end{aligned}$$

Now, if $y \in E^{\times}(M)$ and $x \in E(M)$ then

$$\tau(x E_n^- E_n^+(y)) = \tau(E_n^- E_n^+(x) y) \underset{n \to \infty}{\to} \tau(xy).$$

□

Proposition 3.5. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ such that either $E(M)$ has order continuous norm or $E(M) = F^\times(M)$ where $F(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then*

$$\Big\{x \in \cup_{n\geq 1}(L_1 \cap L_\infty)(M_n^- \cap M_n^+) \ : \ \forall n \notin I^-, \ D_n^-(x) = 0, \ \forall n \notin I^+, \ D_n^+(x) = 0\Big\}$$

is a weakly dense subspace of $E^{\mathrm{sub}}(M)$.

Proof. Fix $x \in E^{\mathrm{sub}}(M)$. Then we know that the sequence $(E_n^- E_n^+(x))_{n\geq 1}$ belongs to $E^{\mathrm{sub}}(M)$, and by the previous lemma it converges weakly to x in $E(M)$. Thus, by the Commutative Assumption we can assume that there is $n \geq 1$ such that $E_n^-(x) = E_n^+(x) = x$, so that we have

$$x = \sum_{i,j=1}^{n} D_i^- D_j^+(x) = \sum_{i\in I^-, j\in I^+, i,j\leq n} D_i^- D_j^+(x).$$

As $(L_1 \cap L_\infty)(M)$ is weakly dense in $E(M)$, there is a net $(y_\alpha)_\alpha$ of $(L_1 \cap L_\infty)(M)$ that converges weakly to x in $E(M)$. We set

$$x_\alpha := \sum_{i\in I^-, j\in I^+, i,j\leq n} D_i^- D_j^+(y_\alpha).$$

Then $x_\alpha \in (L_1 \cap L_\infty)(M_n)$ with $D_n^-(x) = 0$ for all $n \notin I^-$ and $D_n^+(x) = 0$ for all $n \notin I^+$. As the net $(x_\alpha)_\alpha$ converges weakly to x in $E(M)$, the proof is complete. $\square$

As in the previous paragraph, from the above proposition we derive the following result.

Theorem 3.6. *Let $1 < p, q \leq \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(M) := K_\Phi(L_p(M), L_q(M))$ has order continuous norm. Then*

$$E^{\mathrm{sub}}(M) = K_\Phi(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$$

with equivalent norms, with constants depending on p, q only.

By considering in particular the real interpolation functors, we deduce the following result.

Corollary 3.7. *The compatible family $(L_p^{\mathrm{sub}}(M))_{p\in(1,\infty]}$ is a real interpolation scale.*

3.3 Theorem C

Let M, N be two tracial von Neumann algebras respectively equipped with filtrations $(M_n)_{n\geq 1}$, $(N_n)_{n\geq 1}$ with associated conditional expectations denoted by $(E_n)_{n\geq 1}$, $(F_n)_{n\geq 1}$. Let $O := M\bar{\otimes}N$ denote the tensor product tracial von Neumann algebra. Let $(O_n^-)_{n\geq 1}$, $(O_n^+)_{n\geq 1}$ be the two filtrations on O such that, for $n \geq 1$ we have

$$\begin{aligned} O_{2n-1}^- &:= M_n\bar{\otimes}N_n, & O_{2n}^- &:= M_{n+1}\bar{\otimes}N_n, \\ O_{2n-1}^+ &:= M_n\bar{\otimes}N_{n+1}, & O_{2n}^+ &:= M_{n+1}\bar{\otimes}N_{n+1}. \end{aligned} \tag{3.3}$$

Let $(E_n^\pm)_{n\geq 1}$ and $(D_n^\pm)_{n\geq 1}$ respectively denote the conditional expectations and increment projections associated with the filtration $(O_n^\pm)_{n\geq 1}$. Thus, for $n \geq 1$ we have

$$\begin{aligned} E_{2n-1}^- &:= E_n\bar{\otimes}F_n, & E_{2n}^- &:= E_{n+1}\bar{\otimes}F_n, \\ E_{2n-1}^+ &:= E_n\bar{\otimes}F_{n+1}, & E_{2n}^+ &:= E_{n+1}\bar{\otimes}F_{n+1}. \end{aligned}$$

Lemma 3.8. *The two filtrations $(O_n^-)_{n\geq 1}$ and $(O_n^+)_{n\geq 1}$ satisfy the two following conditions.*

Commutativity condition. *For every $m, n \geq 1$, we have*

$$E_m^- E_n^+ = E_n^+ E_m^-.$$

Orthogonality condition. *For every $m, n \geq 1$, we have*

$$D_{2m}^- D_{2n-1}^+ = D_{2n-1}^+ D_{2m}^- = 0.$$

Proof. It is clear that the Commutativity condition holds. Now we check the Orthogonality condition. Fix $m \geq 1$. If $n \geq 2$, then by an easy computation we have

$$D_{2m}^- D_{2n-1}^+ = E_n(E_{m+1} - E_m) \otimes F_m(F_{n+1} - F_n)$$

If $m \leq n$, then $F_m(F_{n+1}-F_n) = 0$ and if $m \geq n$, then $E_n(E_{m+1}-E_m) = 0$. Thus, in all cases, the above is zero. Finally, if $n = 1$, then $D_{2m}^- D_1^+ = E_1(E_{m+1}-E_m)\otimes F_m = 0$. □

As a result, if we set $I^- := \{2n-1 \ : \ n \geq 1\}$ and $I^+ = \{2n \ : \ n \geq 1\}$, then we are exactly in the setting in the previous paragraph. In coherence with this remark, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$ we set

$$E^{\mathrm{sub}}(O) := \Big\{x \in E(O) \ : \ \forall n \geq 1,\ D_{2n}^-(x) = 0,\ \forall n \geq 1,\ D_{2n-1}^+(x) = 0\Big\}.$$

As before, $E^{\mathrm{sub}}(O)$ is a weakly closed subspace of $E(O)$ w.r.t. Köthe duality, and in addition it is stabilised by $E_n^\pm$ for every $n \geq 1$. Moreover, Theorem B applies in the present context, so that, if $1 < p, q \leq \infty$, then the subcouple $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$ is K-closed in the compatible couple $(L_p(O), L_q(O))$. The third main result of the paper extends this result for every $1 \leq p, q \leq \infty$.

Theorem C. *If $1 \le p, q \le \infty$, then the subcouple $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*

As in the previous paragraph, from Theorem C we deduce the following result.

Theorem 3.9. *Let $1 \le p, q \le \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*

$$E^{\mathrm{sub}}(O) = K_\Phi(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$$

with equivalent norms, with universal constants.

By considering in particular the real interpolation functors, we deduce the following result.

Corollary 3.10. *The compatible family $(L_p^{\mathrm{sub}}(O))_{p\in[1,\infty]}$ is a real interpolation scale.*

4 Proofs of the main results

4.1 The tools

4.1.1 Admissible operators

In this paragraph M is a von Neumann algebra equipped with a (n.s.f.) trace τ.

Definition 4.1. A bounded operator $T : L_2(M) \to L_2(M)$ is *admissible* with constant $C > 0$ if for every $y \in (L_1 \cap L_2)(M)$ and $\lambda > 0$ there is a decomposition $y = a + b + c$ with $a, b, c \in L_2(M)$ and a projection $p \in M$ such that

- $\|T(a)\|_2^2 \leq C^2\lambda\|y\|_1$,
- $\|T(b)\|_1 \leq C\|y\|_1$,
- $\tau(1 - p) \leq C\lambda^{-1}\|y\|_1$,
- $pT(c)p = 0$.

Remark 4.2. Note that every bounded operator T on $L_2(M)$ which is also L_1-bounded is clearly admissible with constant $\|T\|_{L_1 \to L_1}$.

The case of bounded admissible operators which are idempotent is of particular importance in this paper because of the following result. The proof of which is inspired by the proof of [2][Lemma 2.4].

Theorem 4.3. *Let $P : L_2(M) \to L_2(M)$ be a bounded idempotent operator which is admissible with constant $C > 0$. Let A_1 and A_2 denote the closure of $L_1(M) \cap P(L_2(M))$ in $L_1(M)$ and $L_2(M)$ respectively. Then the subcouple (A_1, A_2) is K-closed in $(L_1(M), L_2(M))$ with a constant depending on C and $\|P\| := \|P\|_{L_2 \to L_2}$ only.*

Proof. Fix $t > 0$. Let $x \in L_1(M) \cap P(L_2(M))$. Let $y, z \in (L_1 + L_2)(M)$ such that $x = y + z$ and $\|y\|_1 + t\|z\|_2 \leq 1$. Then $y \in (L_1 \cap L_2)(M)$ and applying the definition with y and the parameter $\lambda = t^{-2}$, there is a decomposition $y = a + b + c$ with $a, b, c \in L_2(M)$ and a projection $p \in M$ such that

- $\|P(a)\|_2^2 \leq C^2t^{-2}\|y\|_1 \leq C^2t^{-2}$,
- $\|P(b)\|_1 \leq C\|y\|_1 \leq C$,
- $\tau(1 - p) \leq Ct^2\|y\|_1 \leq Ct^2$,
- $pP(c)p = 0$.

Then, we set

$$y' := P(b+c), \qquad z' := P(a+z).$$

As $x = P(x) = P(y) + P(z)$, it is clear that $x = y' + z'$. On the one hand, we have

$$\|z'\|_2 \leq \|P(a)\|_2 + \|P\|\|z\|_2 \leq Ct^{-1} + \|P\|t^{-1} = (C + \|P\|)t^{-1}$$

On the other hand, we can write $y' = py'p + (1-p)y'p + py'(1-p)$. As $py'p = pP(b+c)p = pP(b)p$, we get

$$\begin{aligned}
\|y'\|_1 &\leq \|py'p\|_1 + \|(1-p)y'p\|_1 + \|py'(1-p)\|_1 \\
&\leq \|py'p\|_1 + \|(1-p)y'\|_1 + \|y'(1-p)\|_1 \\
&= \|pP(b)p\|_1 + \|(1-p)u'\|_1 + \|u'(1-p)\|_1 \\
&= \|P(b)\|_1 + \|(1-p)u'\|_1 + \|u'(1-p)\|_1 \\
&\leq C + \|(1-p)u'\|_1 + \|u'(1-p)\|_1.
\end{aligned}$$

As $y' = x - z' = y + (z - z')$ we have

$$\begin{aligned}
\|(1-p)y'\|_1 &\leq \|(1-p)y\|_1 + \|(1-p)(z-z')\|_2 \\
&\leq \|y\|_1 + \|1-p\|_2\|z-z'\|_2 \\
&\leq \|y\|_1 + \tau(1-p)^{1/2}(\|z\|_2 + \|z'\|_2) \\
&\leq 1 + \tau(1-p)^{1/2}(t^{-1} + (C+\|P\|)t^{-1}) \\
&\leq 1 + C^{1/2}t(t^{-1} + (C+\|P\|)t^{-1}) \\
&= 1 + C^{1/2}(1 + C + \|P\|) =: C',
\end{aligned}$$

and similarly, we have

$$\|y'(1-p)\|_1 \leq C'.$$

Thus

$$\|y'\|_1 \leq C + 2C'.$$

As a result, we finally get

$$\|y'\|_1 + t\|z'\|_2 \leq C + 2C' + C + \|P\| := C''.$$

As clearly $y', z' \in L_1(M) \cap P(L_2(M))$, this shows that

$$K_t(x, A_1, A_2) \leq C'' K_t(x, L_1(M), L_2(M)).$$

By definition $L_1(M) \cap P(L_2(M))$ is dense both A_1 and A_2, thus, it is dense in $A_1 + A_1$. As the K-functional $K_t(-, A_1, A_2)$ is continuous on $A_1 + A_2$ and the K-functionnal $K_t(-, L_1(M), L_2(M))$ is continuous on $(L_1 + L_2)(M)$, so also on $A_1 + A_2$, it follows that the above estimate extends to $x \in A_1 + A_2$. The proof is complete. $\square$

4.1.2 Weakly admissible idempotent operators

The definition of admissible operator has the major drawback that, in general, neither the sum nor the composition of two bounded admissible operators are admissible. At the end we will obtain a similar general K-closed result for these operators. In this subsection we introduce a more flexible definition for bounded idempotents operator. Again, in this paragraph M is a von Neumann algebra equipped with a (n.s.f.) trace denoted τ.

Definition 4.4. A bounded idempotent operator $P : L_2(M) \to L_2(M)$ is *weakly admissible* on a subspace D of $(L_1 \cap L_2)(M)$ with constant $C > 0$ if $P(D) \subset (L_1 \cap L_2)(M)$ and if for every $y \in D$ and $\lambda > 0$ there is a decomposition $P(y) = a + b + c$ with $a, b, c \in P(L_2(M))$ and a projection $p \in M$ such that

- $\|a\|_2^2 \leq C^2 \lambda \|y\|_1$,
- $\|b\|_1 \leq C\|y\|_1$,
- $\tau(1-p) \leq C\lambda^{-1}\|y\|_1$,
- $pcp = 0$.

Remark 4.5. It is clear that every bounded idempotent operator P on $L_2(M)$ which is admissible is weakly admissible on $\{y \in (L_1 \cap L_2)(M) \ : \ P(y) \in (L_1 \cap L_2)(M)\}$ with same constant.

Proposition 4.6. *Let $P^-, P^+ : L_2(M) \to L_2(M)$ be two bounded idempotent operators which are weakly admissible on subspaces D^-, D^+ of $(L_1 \cap L_2)(M)$ with constant $C_- > 0, C_+ > 0$ respectively, such that P^-, P^+ are orthogonal, i.e. $P^-P^+ = P^+P^- = 0$. Then the bounded idempotent operator $P := P^- + P^+$ is weakly admissible on $D := D^- \cap D^+$.*

Proof. Let $y \in D$ and $\lambda > 0$. There is a decomposition $P^\pm(y) = a^\pm + b^\pm + c^\pm$ with $a, b, c \in P^\pm(L_2(M))$ and a projection $p^\pm \in M$ such that

- $\|a^\pm\|_2^2 \leq C_\pm^2 \lambda \|y\|_1$,
- $\|b^\pm\|_1 \leq C_\pm\|y\|_1$,
- $\tau(1-p^\pm) \leq C_\pm\lambda^{-1}\|y\|_1$,
- $p^\pm c^\pm p^\pm = 0$.

We set

$$a := a^- + a^+, \qquad b := b^- + b^+, \qquad c := c^- + c^+, \qquad p := p^- \wedge p^+.$$

Then clearly $P(y) = a + b + c$, with $a, b, c \in P(L_2(M))$, and

- $\|a\|_2^2 \leq (\|a^-\|_2 + \|a^+\|_2)^2 \leq (C_- + C_+)^2\lambda\|y\|_1$,
- $\|b\|_1 = \|b^- + b^+\|_1 \leq \|b^-\|_1 + \|b^+\|_1 \leq (C_- + C_+)\|y\|_1$,
- $\tau(1-p) = \tau((1-p^-) \vee (1-p^+)) \leq \tau(1-p^-) + \tau(1-p^+) \leq (C_- + C_+)\lambda^{-1}\|y\|_1$,
- $pcp = pc^-p + pc^+p = 0$.

The proof is complete. □

Lemma 4.7. *Let $P : L_2(M) \to L_2(M)$ be a bounded idempotent operator which is weakly admissible on a subspace D of $(L_1 \cap L_2)(M)$ with constant $C > 0$. Let $x \in L_1(M) \cap P(L_2(M))$ and $t > 0$. Let $y, z \in (L_1 + L_2)(M)$ such that $x = y + z$ with $y \in D$ and $\|y\|_1 + t\|z\|_2 \leq 1$. Then there is $y', z' \in P(L_2(M))$ such that $x = y' + z'$ with $\|y'\|_1 + t\|z'\|_2 \leq C'$ where $C' > 0$ depends on C only.*

Proof. It suffices to mimic the proof of Theorem 4.8. □

Theorem 4.8. *Let $P : L_2(M) \to L_2(M)$ be a bounded idempotent operator and D be a subset of $(L_1 \cap L_2)(M)$ such that*

1. *P is weakly admissible on D with constant $C > 0$.*
2. *For every $x \in D \cap P(L_2(M))$, there is a contractive compatible operator*

$$E_x : (L_1(M), L_2(M)) \to (L_1(M), L_2(M))$$

such that $E_x(x)$ and

$$E_x((L_1 \cap L_2)(M)) \subset D, \qquad E_x(P(L_2(M)) \subset P(L_2(M)).$$

Let A_1 and A_2 denote the closure of $D \cap P(L_2(M))$ in $L_1(M)$ and $L_2(M)$ respectively. Then the subcouple (A_1, A_2) is K-closed in $(L_1(M), L_2(M))$ with a constant depending on C and $\|P\| := \|P\|_{L_2 \to L_2}$ only.

Proof of Theorem 4.8. Fix $t > 0$. Let $x \in D \cap P(L_2(M))$. Let $y, z \in (L_1 + L_2)(M)$ such that $x = y + z$ and $\|y\|_1 + t\|z\|_2 \leq 1$. Then $x = E_x(x) = E_x(y) + E_x(z)$. Moreover, we have $E_x(y) \in D$, and $\|E_x(y)\|_1 + t\|E_x(z)\|_2 \leq \|y\|_1 + t\|z\|_2 \leq 1$. Thus, by the previous lemma there is $y', z' \in P(L_2(M))$ such that $x = y' + z'$ and $\|y'\|_1 + t\|z'\|_2 \leq C'$ where $C' > 0$ depends on C and $\|P\|$ only. Then $x = E_x(x) = E_x(y') + E_x(z')$ and $\|E_x(y')\|_1 + t\|E_x(z')\|_2 \leq \|y'\|_1 + t\|z'\|_2 \leq C'$. Moreover, by hypothesis we have $E_x(y'), E_x(y') \in D \cap P(L_2(M))$. This shows that

$$K_t(x, A_1, A_2) \leq C' K_t(x, L_1(M), L_2(M)).$$

Finally, the above estimate extends to $x \in A_1 + A_2$ because $D \cap P(L_2(M))$ is dense in both A_1 and A_2 by definition. $\square$

4.1.3 Martingale transforms

Let M be a von Neumann algebra equipped with a (n.s.f.) trace τ and a filtration $(M_n)_{n\geq 1}$ with associated conditional expectations denoted $(E_n)_{n\geq 1}$ and associated increment projections denoted $(D_n)_{n\geq 1}$.

Let $(a_n)_{n\geq 1}$ be a bounded sequence of scalars. The associated *martingale transform* is defined as

$$T : \begin{cases} L_2(M) & \to & L_2(M) \\ x & \mapsto & \sum_{n\geq 1} a_n D_n(x) \end{cases}$$

The martingale transform T is clearly well-defined and L_2-bounded with

$$\|T\|_{L_2 \to L_2} \leq \sup_{n\geq 1} |a_n|.$$

The following result is proved in [11].

Theorem 4.9. *For every $1 < p < \infty$, the martingale transform T is L_p-bounded, with*

$$\|T\|_{L_p \to L_p} \leq C \frac{p^2}{p-1} \sup_{n\geq 1} |a_n|,$$

where $C > 0$ is a universal constant.

Theorem 4.10. *The martingale transform $T : L_2(M) \to L_2(M)$ is admissible with a constant depending on $\sup_{n\geq 1} |a_n|$ only.*

Proof. Let $y \in (L_1 \cap L_2)(M)$ and $\lambda > 0$. By Gundy's decomposition theorem (Theorem 2.30) there is a decomposition $y = a + b + c$ with $a, b, c \in L_2(M)$ and a projection $p \in M$ such that

- $\|a\|_2^2 \leq C^2 \lambda \|y\|_1$,
- $\sum_{n\geq 1} \|D_n(b)\|_1 \leq C\|y\|_1$,
- $\tau(1-p) \leq C\lambda^{-1}\|y\|_1$,
- $pD_n(c)p = 0$ for every $n \geq 1$.

where $C > 0$ is a universal constant. Then

- $\|T(a)\|_2^2 \leq \|T\|_{L_2 \to L_2}^2 \|a\|_2^2 \leq \sup_{n\geq 1} |a_n|^2 C^2 \lambda \|y\|_1$,

- $\|T(b)\|_1 = \left\| \sum_{n\geq 1} a_n D_n(b) \right\|_1 \leq \sup_{n\geq 1} |a_n| \sum_{n\geq 1} \|D_n(b)\|_1 \leq C \sup_{n\geq 1} |a_n| \|y\|_1$,
- $pT(c)p = \sum_{n\geq 1} a_n p D_n(c) p = 0$.

This concludes the proof. $\square$

The previous result extends in a more general setting. Let $(T_n)_{n\geq 1}$ be a sequence of bounded operators on $L_2(M)$ such that T_n is M_{n-1}-linear for every $n \geq 2$. The associated *generalised martingale transform* is defined as

$$T : \begin{cases} L_2(M) & \to & L_2(M) \\ x & \mapsto & \sum_{n\geq 1} T_n(D_n(x)) \end{cases}$$

In order to ensure that T is well-defined, we will assume that the serie $\sum_{n\geq 1} T_n(D_n(x))$ converges in $L_2(M)$ if $x \in L_2(M)$. We will also assume that T is L_2-bounded, and that

$$\sup_{n\geq 1} \|T_n\|_{L_1 \to L_1} < \infty.$$

Theorem 4.11. *The generalised martingale transform $T : L_2(M) \to L_2(M)$ is admissible with a constant depending on $\|T\|_{L_2\to L_2}$ and $\sup_{n\geq 1} \|T_n\|_{L_1\to L_1} < \infty$ only.*

Proof. Let $y \in (L_1 \cap L_2)(M)$ and $\lambda > 0$. By Gundy's decomposition theorem (Theorem 2.30) there is a decomposition $y = a + b + c$ with $a, b, c \in L_2(M)$ and an adapted sequence $(p_n)_{n\geq 1}$ of projections of M such that

- $\|a\|_2^2 \leq C^2 \lambda \|y\|_1$,
- $\sum_{n\geq 1} \|D_n(b)\|_1 \leq C\|y\|_1$,
- $\tau(1 - p) \leq C\lambda^{-1}\|y\|_1$ where $p := \wedge_{n\geq 1} p_n$,
- $p_{n-1} D_n(c) p_{n-1} = 0$ for every $n \geq 1$ (with the convention $p_0 = 1$).

where $C > 0$ is a universal constant. We have

$$\|T(a)\|_2^2 \leq \|T\|_{L_2\to L_2}^2 \|a\|_2^2 \leq \|T\|_{L_2\to L_2}^2 C^2 \lambda \|y\|_1$$

and

$$\|T(b)\|_1 = \left\| \sum_{n\geq 1} T_n(D_n(b)) \right\|_1 \leq \sup_{n\geq 1} \|T_n\|_{L_1\to L_1} \sum_{n\geq 1} \|D_n(b)\|_1 \leq \sup_{n\geq 1} \|T_n\|_{L_1\to L_1} C \|y\|_1$$

Now fix $n \geq 2$. As T_n is M_{n-1}-linear and $p_{n-1} \in M_{n-1}$, we have

$$p_{n-1} T_n(D_n(c)) p_{n-1} = T_n(p_{n-1} D_n(c) p_{n-1}) = 0.$$

Thus $pT_{n-1}(D_n(c))p = 0$ for every $n \geq 1$ and

$$pT(c)p = \sum_{n\geq 1} pT_n(D_n(c))p = 0.$$

The proof is complete. $\square$

4.2 The proofs

In this paragraph, we place ourselves within the frameworks introduced in the previous section.

4.2.1 Theorem A

Let M be a tracial von Neumann algebra equipped with a filtration $(M_n)_{n\geq 1}$ with associated conditional expectations denoted $(E_n)_{n\geq 1}$ and associated increment projections denoted $(D_n)_{n\geq 1}$. Let I be a fixed set of positive integer. For $1 \leq p \leq \infty$, we set

$$L_p^{\text{sub}}(M) := \Big\{x \in L_p(M) \ : \ \forall n \notin I,\ D_n(x) = 0\Big\}.$$

It is clear that $L_p^{\text{sub}}(M)$ is a weakly closed subspace of $L_p(M)$ w.r.t. Köthe duality, and in addition it is stabilised by E_n for every $n \geq 1$. Moreover, as a consequence of Proposition 3.5, we know that $L_1^{\text{sub}}(M) \cap L_\infty^{\text{sub}}(M)$ is a weakly dense subspace of $L_p^{\text{sub}}(M)$ w.r.t. Köthe duality, for $1 \leq p \leq \infty$.

The goal of this section is to prove Theorem A, whose statement is recalled below.

Theorem 4.12 (Theorem A). *If $1 \leq p, q \leq \infty$ the subcouple $(L_p^{\text{sub}}(M), L_q^{\text{sub}}(M))$ is K-complemented in the compatible couple $(L_p(M), L_q(M))$ with a universal constant.*

By using the real interpolation machinary introduced in the preliminary section, namely the K-reiteration theorem and the K-Wolff interpolation theorem, we see that Theorem A is a consequence of the four facts stated below.

Fact 1. *Let $1 \leq p \neq q \leq \infty$ such that the subcouple $(L_p^{\text{sub}}(M), L_q^{\text{sub}}(M))$ is K-closed in $(L_p(M), L_q(M))$. Then for every $0 < \theta < 1$, we have $(L_p^{\text{sub}}(M), L_q^{\text{sub}}(M))_{\theta,r} = L_r^{\text{sub}}(M)$ with equivalent norms, with constants depending on p, q only, where $1/r = (1-\theta)/p + \theta/q$.*

Fact 2. *Let $1 < p, q < \infty$. Then the subcouple $(L_p^{\text{sub}}(M), L_q^{\text{sub}}(M))$ is complemented and in particular it is K-complemented in $(L_p(M), L_q(M))$ with a constant depending on p, q only.*

Fact 3. *The subcouple* $(L_1^{\rm sub}(M), L_2^{\rm sub}(M))$ *is* K*-complemented in* $(L_1(M), L_2(M))$ *with a universal constant.*

Fact 4. *The subcouple* $(L_2^{\rm sub}(M), L_\infty^{\rm sub}(M))$ *is* K*-complemented in* $(L_2(M), L_\infty(M))$ *with a universal constant.*

For the proof of the above facts, we need a couple of lemmas.

Lemma 4.13. *Let* $1 \le p \le \infty$*. Let* $L_p^{\rm ort}(M)$ *denote the orthogonal of* $L_q^{\rm sub}(M)$ *in* $L_p(M)$ *w.r.t. Köthe duality, where* $1 \le q \le \infty$ *is such that* $1/p + 1/q = 1$*. Then*

$$L_p^{\rm ort}(M) = \Big\{ x \in L_p(M) \ : \ \forall n \in I,\ D_n(x) = 0 \Big\}.$$

Proof. If $x \in L_p(M)$ is such that $D_n(x) = 0$ for every $n \in I$, then for $y \in L_q^{\rm sub}(M)$, we have

$$\tau(xy) = \sum_{n \ge 1} \tau(D_n(x) D_n(y)) = 0.$$

In the converse way, if $x \in L_p^{\rm ort}(M)$, and if $n \in I$, then for every $y \in L_q(M)$ we clearly have $D_n(y) \in L_q^{\rm sub}(M)$ so that

$$\tau(D_n(x) y) = \tau(x D_n(y)) = 0$$

and as a consequence $D_n(x) = 0$, as desired. □

The proof of the following last lemma is straightforward.

Lemma 4.14. *Let* P *denote the orthogonal projection on* $L_2(M)$ *onto* $L_2^{\rm sub}(M)$*. Then for every* $x \in L_2(M)$*, we have*

$$P(x) = \sum_{n \in I} D_n(x), \qquad (I - P)(x) = \sum_{n \notin I} D_n(x), \quad \text{in } L_2(M).$$

In particular, P *and* $I - P$ *are martingale transforms.*

Now we turn to the proof of Facts 1-4.

Proof of Fact 1. Let $0 < \theta < 1$. As $(L_p(M), L_q(M))_{\theta,r} = L_r(M)$ with equivalent norms, with constants depending on p, q only, and because $(L_p^{\rm sub}(M), L_q^{\rm sub}(M))_{\theta,r}$ is K-closed in $(L_p(M), L_q(M))_{\theta,r}$ with a universal constant, we know that we have an inclusion operator

$$(L_p^{\rm ad}(M), L_q^{\rm ad}(M)) \to L_r(M)$$

which is an embedding of normed spaces, with constants depending on p, q only, and with range $(L_p^{\mathrm{ad}}(M) + L_q^{\mathrm{ad}}(M)) \cap L_r(M)$. Thus, it suffices to show that $(L_p^{\mathrm{ad}}(M) + L_q^{\mathrm{ad}}(M)) \cap L_r(M)$ is a dense subspace of $L_r^{\mathrm{ad}}(M)$. It is clear that is is indeed a subspace, and in addition is contains $L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M)$ which is known to be weakly-dense in $L_r(M)$ w.r.t Köthe duality. As $r < \infty$, it is norm-dense in $L_r(M)$. The proof is complete. $\square$

Proof of Fact 2. Let $1 < p < \infty$. From Lemma 4.14 and Theorem 4.9 we deduce that the orthogonal projection P on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ is L_p-bounded with $\|P\|_{L_p \to L_p} \leq C_p$ where $C_p > 0$ is a constant depending on p only. Moreover, we know that $L_2^{\mathrm{sub}}(M) \cap L_p(M)$ is a norm-dense subspace in $L_p^{\mathrm{sub}}(M)$ as it contains $L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M)$, which implies that $L_p^{\mathrm{sub}}(M)$ must coincide with the range of the bounded idempotent operator $L_p(M) \to L_p(M)$ induced by P. It directly follows that, for $1 < p, q < \infty$, the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is complemented in $(L_p(M), L_q(M))$ with a constant depending on p, q only. $\square$

Proof of Fact 3. From Lemma 4.14 and Theorem 4.10 we deduce that the orthogonal projection P on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ is admissible with a universal constant. Moreover, we know that $L_1(M) \cap L_2^{\mathrm{sub}}(M)$ is a norm-dense subspace of both $L_1^{\mathrm{sub}}(M)$ and $L_2^{\mathrm{sub}}(M)$ as it contains $L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M)$. By Theorem 4.3, we deduce that the subcouple $(L_1^{\mathrm{sub}}(M), L_2^{\mathrm{sub}}(M))$ is K-closed in $(L_1(M), L_2(M))$ with a universal constant. As we clearly have $(L_1^{\mathrm{sub}}(M) + L_2^{\mathrm{sub}}(M)) \cap L_1(M) = L_1^{\mathrm{sub}}(M)$ and $(L_1^{\mathrm{sub}}(M) + L_2^{\mathrm{sub}}(M)) \cap L_2(M) = L_2^{\mathrm{sub}}(M)$, the desired conclusion follows. $\square$

Proof of Fact 4. By applying Fact 3 with the complement subset of I instead of I, and by taking into account Lemma 4.13, we deduce that the subcouple $(L_1^{\mathrm{ort}}(M), L_2^{\mathrm{ort}}(M))$ is K-closed in $(L_1(M), L_2(M))$ with a universal constant. As the compatible couple $(L_1(M), L_2(M))$ is regular, by Pisier's duality lemma we deduce that the subcouple $(L_2^{\mathrm{sub}}(M), L_\infty^{\mathrm{sub}}(M))$ is K-complemented in $(L_2(M), L_\infty(M))$ with a universal constant. $\square$

4.2.2 Theorem B

Let M be a tracial von Neumann algebra equipped with two filtrations $(M_n^-)_{n\geq 1}$, $(M_n^+)_{n\geq 1}$ with associated conditional expectations denoted by $(E_n^-)_{n\geq 1}$, $(E_n^+)_{n\geq 1}$ and increment projections denoted by $(D_n^-)_{n\geq 1}$, $(D_n^+)_{n\geq 1}$ respectively. Let I^-, I^+ be two fixed sets of positive integers. We add the following two assumptions.

Commutation Assumption. The filtrations $(M_n^-)_{n\geq 1}$, $(M_n^+)_{n\geq 1}$ *commute* in the sense that for every $m,n\geq 1$, we have

$$E_m^- E_n^+ = E_n^+ E_m^-. \tag{4.1}$$

Orthogonality Assumption. For every $m\notin I^-$ and $n\notin I^+$, we have

$$D_m^- D_n^+ = D_n^+ D_m^- = 0. \tag{4.2}$$

For $1\leq p\leq\infty$, as in the previous section we set

$$(L_p)_{\pm}^{\mathrm{sub}}(M) := \Big\{ x\in L_p(M) \ : \ \forall n\notin I^{\pm},\ D_n^{\pm}(x)=0\Big\},$$

and finally we set

$$\begin{aligned} L_p^{\mathrm{sub}}(M) &:= (L_p)_-^{\mathrm{sub}}(M)\cap (L_p)_+^{\mathrm{sub}}(M) \\ &= \Big\{ x\in L_p(M) \ : \ \forall n\notin I^-,\ D_n^-(x)=0,\ \forall n\notin I^+,\ D_n^+(x)=0\Big\}. \end{aligned}$$

It is clear that $L_p^{\mathrm{sub}}(M)$ is a weakly closed subspace of $L_p(M)$ w.r.t. Köthe duality, and in addition it is stabilised by $E_n^{\pm}$ for every $n\geq 1$. Moreover, as a consequence of Proposition 3.5, we know that $L_1^{\mathrm{sub}}(M)\cap L_\infty^{\mathrm{sub}}(M) = L_1(M)\cap L_\infty^{\mathrm{sub}}(M) = L_1^{\mathrm{sub}}(M)\cap L_\infty(M)$ is a weakly dense subspace of $L_p^{\mathrm{sub}}(M)$ w.r.t. Köthe duality, for $1\leq p\leq\infty$.

The goal of this section is to prove Theorem B, whose statement is recalled below.

Theorem 4.15 (Theorem B). *If $1<p,q\leq\infty$ the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-complemented in $(L_p(M), L_q(M))$ with a constant depending on p,q only.*

Again, by using the real interpolation machinary introduced in the preliminary section, namely the K-reiteration theorem and the K-Wolff interpolation theorem, we see that Theorem B is a consequence of the four facts stated below.

Fact 5. *Let $1\leq p\neq q\leq\infty$ such that the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in $(L_p(M), L_q(M))$. Then for every $0<\theta<1$, we have $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))_{\theta,r} = L_r^{\mathrm{sub}}(M)$ with equivalent norms, with constants depending on p,q only, where $1/r=(1-\theta)/p+\theta/q$.*

Fact 6. *Let $1<p,q<\infty$. Then the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is complemented and in particular it is K-complemented in $(L_p(M), L_q(M))$ with a constant depending on p,q only.*

Fact 7. *The subcouple $(L_2^{\mathrm{sub}}(M), L_\infty^{\mathrm{sub}}(M))$ is K-complemented in $(L_2(M), L_\infty(M))$ with a universal constant.*

For the proof of the above facts, we need a couple of lemmas.

Lemma 4.16. *Let $1 \leq p \leq \infty$. Let $L_p^{\mathrm{ort}}(M)$ denote the orthogonal of $L_q^{\mathrm{sub}}(M)$ in $L_p(M)$ w.r.t. Köthe duality, where $1 \leq q \leq \infty$ is such that $1/p + 1/q = 1$. Then*

$$L_p^{\mathrm{ort}}(M) = \Big\{ x \in L_p(M) \ : \ \forall y \in L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M), \ \tau(xy) = 0 \Big\}.$$

Moreover, $L_p^{\mathrm{ort}}(M)$ is stabilised by $E_n^{\pm}$ for every $n \geq 1$. Finally, $L_1^{\mathrm{ort}}(M) \cap L_\infty^{\mathrm{ort}}(M) = L_1^{\mathrm{ort}}(M) \cap L_\infty(M) = L_1^{\mathrm{ort}}(M) \cap L_\infty(M)$ is a weakly dense subspace of $L_p(M)$ w.r.t Köthe duality.

Proof. The first assertion of the lemma follows from the fact that $L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M)$ is weakly dense in $L_p(M)$. The second assertion of the lemma follows from the fact that $L_p^{\mathrm{sub}}(M)$ is stabilised by $E_n^{\pm}$ for every $n \geq 1$, combined with the fact that $E_n^{\pm}$ is self-adjoint, for every $n \geq 1$. Now we turn to the proof of the last assertion. We set

$$(L_p)_{\pm}^{\mathrm{ort}}(M) := \Big\{ x \in L_p(M) \ : \ \forall n \in I^{\pm}, \ D_n^{\pm}(x) = 0 \Big\}.$$

As we know that $(L_p)_{\pm}^{\mathrm{ort}}(M)$ is the orthogonal of $(L_q)_{\pm}^{\mathrm{sub}}(M)$ in $L_p(M)$, and because $L_q^{\mathrm{sub}}(M) = (L_q)_{-}^{\mathrm{sub}}(M) \cap (L_q)_{+}^{\mathrm{sub}}(M)$, we directly deduce that $L_p^{\mathrm{ort}}(M)$ is the weak-closure of $(L_p)_{-}^{\mathrm{ort}}(M) + (L_p)_{+}^{\mathrm{ort}}(M)$ in $L_p(M)$. Besides, we already know that $(L_1)_{\pm}^{\mathrm{ort}}(M) \cap (L_\infty)_{\pm}^{\mathrm{ort}}(M)$ is a weakly dense subspace of $(L_p)_{\pm}^{\mathrm{ort}}(M)$ in $L_p(M)$. Thus, we find that

$$(L_1)_{-}^{\mathrm{ort}}(M) \cap (L_\infty)_{-}^{\mathrm{ort}}(M) + (L_1)_{+}^{\mathrm{ort}}(M) \cap (L_\infty)_{+}^{\mathrm{ort}}(M)$$

is a weakly dense subspace of $L_p^{\mathrm{ort}}(M)$ in $L_p(M)$. Hence, to conclude, it suffices to check that $L_1^{\mathrm{ort}}(M) \cap L_\infty^{\mathrm{ort}}(M)$ contains

$$(L_1)_{-}^{\mathrm{ort}}(M) \cap (L_\infty)_{-}^{\mathrm{ort}}(M) + (L_1)_{+}^{\mathrm{ort}}(M) \cap (L_\infty)_{+}^{\mathrm{ort}}(M).$$

But we already know that $L_1^{\mathrm{ort}}(M) \cap L_\infty^{\mathrm{ort}}(M)$ contains

$$\Big[(L_1)_{-}^{\mathrm{ort}}(M) + (L_1)_{+}^{\mathrm{ort}}(M)\Big] \cap \Big[(L_\infty)_{-}^{\mathrm{ort}}(M) + (L_\infty)_{+}^{\mathrm{ort}}(M)\Big]$$

and the latter clearly contains

$$(L_1)_{-}^{\mathrm{ort}}(M) \cap (L_\infty)_{-}^{\mathrm{ort}}(M) + (L_1)_{+}^{\mathrm{ort}}(M) \cap (L_\infty)_{+}^{\mathrm{ort}}(M).$$

The proof is complete. □

Lemma 4.17. *Let P denote the orthogonal projection on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$. Let $P_\pm$ denote the orthogonal projection on $L_2(M)$ onto $(L_2)_\pm^{\mathrm{sub}}(M)$. Then $P = P_- P_+ = P_+ P_-$, the projections $I - P_-$ and $I - P_+$ are orthogonal and $I - P = (I - P^-) + (I - P^+)$. Moreover, if $x \in L_2(M)$, then*

$$P_\pm(x) = \sum_{n \in I^\pm} D_n^\pm(x), \qquad (I - P_\pm)(x) = \sum_{n \notin I^\pm} D_n^\pm(x), \quad \text{in } L_2(M).$$

In particular, $P_\pm$ and $I - P_\pm$ are martingale transforms.

Proof. The expressions for $P_\pm$ and $I - P_\pm$ are obtained easily. From the Commutativity Assumption we easily deduce that $P_- P_+ = P_+ P_-$ is the orthogonal projection onto $(L_2)_-^{\mathrm{sub}}(M) \cap (L_2)_+^{\mathrm{sub}}(M) = L_2^{\mathrm{sub}}(M)$, i.e. $P_- P_+ = P_+ P_- = P$. From the Orthogonality Assumption we also easily deduce that $I - P_-$ and $I - P_+$ are orthogonal. Finally, developping the identity $(I - P^-)(I - P^+) = 0$ yields the identity $I - P = (I - P^+) + (I - P^-)$. □

Now we turn to the proof of Facts 5-7.

Proof of Fact 5. It suffices to mimic the proof of Fact 1 of the previous section. □

Proof of Fact 6. Let $1 < p < \infty$. From Lemma 4.17 and Theorem 4.9 we deduce that the orthogonal projection P on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ is L_p-bounded with $\|P\|_{L_p \to L_p} \leq C_p$ where $C_p > 0$ is a constant depending on p only. Moreover, we know that $L_2^{\mathrm{sub}}(M) \cap L_p(M)$ is a norm-dense subspace of $L_p^{\mathrm{sub}}(M)$ as it contains $L_1^{\mathrm{sub}}(M) \cap L_\infty^{\mathrm{sub}}(M)$, which implies that $L_p^{\mathrm{sub}}(M)$ must coincide with the range of the bounded idempotent operator $L_p(M) \to L_p(M)$ induced by P. It directly follows that, for $1 < p, q < \infty$, the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is complemented in $(L_p(M), L_q(M))$ with a constant depending on p, q only. □

Proof of Fact 7. From Lemma 4.17 and 4.10, we know that $I - P_-$ and $I - P_+$ are admissible with a universal constant. From Proposition 4.6, we deduce that $I - P = (I - P_-) + (I - P_+)$ is weakly admissible on $\{y \in (L_1 \cap L_2)(M),\ (I - P)(y) \in (L_1 \cap L_2)(M)\}$ with a universal constant. Let

$$D := \cup_{n \geq 1} L_1(M_n) \cap L_2(M_n).$$

Then it is clear that D is contained in $\{y \in (L_1 \cap L_2)(M),\ (I - P)(y) \in (L_1 \cap L_2)(M)\}$, so that $I - P$ is weakly admissible on D. The next step is to check the hypothesis of Theorem 4.8. If $x \in D \cap L_2^{\mathrm{ort}}(M)$, then there is $n \geq 1$ such that $E_n(x) = x$ and it is clear that E_n sends $(L_1 \cap L_2)(M)$ into D and we know by Lemma 4.16 that E_n stabilises

$L_2^{\mathrm{ort}}(M)$, as required to apply the aforementioned theorem. Moreover, as $L_1^{\mathrm{ort}}(M)$ and $L_2^{\mathrm{ort}}(M)$ are stabilised by all the $(E_n)_{n\geq 1}$, and because $L_1^{\mathrm{ort}}(M)\cap L_\infty^{\mathrm{ort}}(M)$ is norm-dense in both $L_1(M)$ and $L_2(M)$ by Lemma 4.16, we deduce that $D\cap L_2^{\mathrm{ort}}(M)$ is a norm-dense subspace of L_1^{ort} and L_2^{ort}. As a result, we find that the subcouple $(L_1^{\mathrm{ort}}(M), L_2^{\mathrm{ort}}(M))$ is K-closed in $(L_1(M), L_2(M))$ with a universal constant. As $(L_1(M), L_2(M))$ is regular, by Pisier's duality lemma we deduce that $(L_2^{\mathrm{sub}}(M), L_\infty^{\mathrm{sub}}(M))$ is K-complemented in $(L_2(M), L_\infty(M))$ with a universal constant. The proof is complete. □

4.2.3 Theorem C

Let M, N be two tracial von Neumann algebras respectively equipped with filtrations $(M_n)_{n\geq 1}$, $(N_n)_{n\geq 1}$ with associated conditional expectations denoted by $(E_n)_{n\geq 1}$, $(F_n)_{n\geq 1}$. Let $O := M\bar{\otimes} N$ denote the tensor product tracial von Neumann algebra. Let $(O_n^-)_{n\geq 1}$, $(O_n^+)_{n\geq 1}$ be the two filtrations on O such that, for $n\geq 1$ we have

$$\begin{aligned} O_{2n-1}^- &:= M_n\bar{\otimes} N_n, & O_{2n}^- &:= M_{n+1}\bar{\otimes} N_n, \\ O_{2n-1}^+ &:= M_n\bar{\otimes} N_{n+1}, & O_{2n}^+ &:= M_{n+1}\bar{\otimes} N_{n+1}. \end{aligned} \tag{4.3}$$

Let $(E_n^\pm)_{n\geq 1}$ and $(D_n^\pm)_{n\geq 1}$ respectively denote the conditional expectations and increment projections associated with the filtration $(O_n^\pm)_{n\geq 1}$. Thus, for $n\geq 1$ we have

$$\begin{aligned} E_{2n-1}^- &:= E_n\bar{\otimes} F_n, & E_{2n}^- &:= E_{n+1}\bar{\otimes} F_n, \\ E_{2n-1}^+ &:= E_n\bar{\otimes} F_{n+1}, & E_{2n}^+ &:= E_{n+1}\bar{\otimes} F_{n+1}. \end{aligned}$$

For $1\leq p\leq\infty$, we set

$$L_p^{\mathrm{sub}}(O) := \Big\{x\in L_p(O)\ :\ \forall n\geq 1,\ D_{2n}^-(x)=0,\ \forall n\geq 1,\ D_{2n-1}^+(x)=0\Big\}.$$

It is clear that $L_p^{\mathrm{sub}}(O)$ is a weakly closed subspace of $E(O)$ w.r.t. Köthe duality, and in addition it is stabilised by $E_n^\pm$ for every $n\geq 1$.

The goal of this paragraph is to prove Theorem C, whose statement is recalled below.

Theorem 4.18 (Theorem C). *If $1\leq p,q\leq\infty$ the subcouple $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$ is K-complemented in $(L_p(O), L_q(O))$ with a constant depending on p,q only.*

As noticed before, if we set $I^- := \{2n-1\ :\ n\geq 1\}$ and $I^+ = \{2n\ :\ n\geq 1\}$, then we are exactly in the setting in the previous paragraph. Hence, all the results of previous paragraph also hold in the present setting. In particular, the two following facts hold.

Fact 8. *Let $1 \leq p \neq q \leq \infty$ such that the subcouple $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$ is K-closed in $(L_p(O), L_q(O))$. Then for every $0 < \theta < 1$, we have $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))_{\theta,r} = L_r^{\mathrm{sub}}(O)$ with equivalent norms, with constants depending on p, q only, where $1/r = (1-\theta)/p + \theta/q$.*

Fact 9. *Let $1 \leq p, q < \infty$. Then the subcouple $(L_p^{\mathrm{sub}}(O), L_q^{\mathrm{sub}}(O))$ is complemented and in particular it is K-complemented in $(L_p(O), L_q(O))$ with a constant depending on p, q only.*

By using the usual real interpolation machinary and the above facts, we see that to conclude the proof of C it suffices to show the following additional fact.

Fact 10. *The subcouple $(L_1^{\mathrm{sub}}(M), L_2(M)^{\mathrm{sub}})$ is K-complemented in $(L_1(O), L_2(O))$ with a universal constant.*

For the proof of this fact, we need a couple of lemmas.

Lemma 4.19. *Let P denote the orthogonal projection on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$. Then for $x \in L_2(O)$, we have*

$$P(x) = \sum_{m,n\geq 1} D_{2m-1}^{-} D_{2n}^{+}(x) = \sum_{m,n\geq 1} D_{2n}^{+} D_{2m-1}^{-}(x) \qquad \text{in } L_2(M).$$

Proof. This lemma has been proved in the previous paragraph. □

Lemma 4.20. *If $x \in L_2(O)$, we have*

$$P(x) = \sum_{n\geq 2} T_n(x), \qquad \text{in } L_2(O),$$

where $T_n := (E_n - E_{n-1}) \otimes (F_n - F_{n-1})$ for every $n \geq 2$.

Proof. If $n, m \geq 1$, an easy computation yields

$$D_{2m-1}^{-} D_{2n}^{+} = E_m(E_{n+1} - E_n) \otimes (F_m - F_{m-1}) F_{n+1}$$

But if $m \leq n$, then $E_m(E_{n+1} - E_n) = 0$, and if $m \geq n+2$, then $(F_m - F_{m-1})F_{n+1} = 0$. Thus, $D_{2m-1}^{-} D_{2n}^{+}$ is always 0 except in the case $m = n+1$, in which case it is equal to T_n. By Lemma 4.19, this completes the proof. □

Now, let $(O_n)_{n\geq 1}$ be the filtration on O such that, for $n \geq 1$ we have

$$O_n := M_n \otimes N_n.$$

Let $(D_n)_{n\geq 1}$ the associated increment projections. Thus, for $n \geq 1$ we have

$$D_n = E_n \otimes F_n - E_{n-1} \otimes F_{n-1}$$

(with the convention $E_0 = F_0 = 0$).

Lemma 4.21. *If $x \in L_2(O)$, we have*

$$P(x) = \sum_{n\geq 2} T_n(D_n(x)), \qquad \text{in } L_2(O).$$

Proof. If $n, m \geq 1$, we have

$$\begin{aligned} T_n D_m &= ((E_n - E_{n-1}) \otimes (F_n - F_{n-1}))(E_m \otimes F_m - E_{m-1} \otimes F_{m-1}) \\ &= E_m(E_n - E_{n-1}) \otimes F_m(F_n - F_{n-1}) - E_{m-1}(E_n - E_{n-1}) \otimes F_{m-1}(F_n - F_{n-1}). \end{aligned}$$

The above computation shows that $T_n D_m$ is always 0 except in the case $m = n$. Thus, if $x \in L_2(O)$, by Lemma 4.20 we find

$$P(x) = \sum_{m\geq 1} P(D_m(x)) = \sum_{m\geq 1} \sum_{n\geq 1} T_n(D_m(x)) = \sum_{n\geq 1} T_n(D_n(x)).$$

□

Now we are able to complete the proof of Fact 10.

Proof of Fact 10. If $n \geq 2$, then T_n is clearly O_{n-1}-linear. Moreover, we clearly have $\sup_{n\geq 2} \|T_n\|_{L_1 \to L_1} \leq 2$. Thus, Lemma 4.21 asserts that P is a generelised martingale transform. In particular, Theorem 4.11 applies, and thus we deduce that P is admissible with a universal constant. Moreover, we know that $L_1(M) \cap L_2^{\text{sub}}(M)$ is norm-dense in both $L_1^{\text{sub}}(M)$ and $L_2^{\text{sub}}(M)$ as it contains $L_1^{\text{sub}}(M) \cap L_\infty^{\text{sub}}(M)$. By Theorem 4.3, it follows that the subcouple $(L_1^{\text{sub}}(M), L_2^{\text{sub}}(M))$ is K-closed in $(L_1(M), L_2(M))$ with a universal constant. As we clearly have $(L_1^{\text{sub}}(M) + L_2^{\text{sub}}(M)) \cap L_1(M) = L_1^{\text{sub}}(M)$ and $(L_1^{\text{sub}}(M) + L_2^{\text{sub}}(M)) \cap L_2(M) = L_2^{\text{sub}}(M)$, the desired conclusion follows. □

5 Applications

In this last section, we use the previous material to derive new results in the context of square inequalities for martingales.

The section is organised as follows. In the first part, we provide the mathematical background on column-row-mixed sequence spaces that will be needed subsequently. In the last parts, we obtain various results for column-row-mixed adapted sequence and martingale increment spaces as a consequence of Theorem C.

5.1 Preliminaries

Let M be a von Neumann algebra equipped with a (n.s.f.) trace τ.

5.1.1 $L_p(\ell_2^c)$-spaces and $L_p(\ell_2^r)$-spaces

Let ℓ_2 denote the Hilbert space of square-summable scalar sequences with canonical Hilbert basis $(\delta_k)_{k\geq 1}$, and let N denote the von Neunamm algebra of all bounded operators on ℓ_2 equipped with its canonical trace denoted tr. For $i, j \geq 1$, let $e_{ij} \in N$ denote the elementary operator such that $e_{ij}\delta_k = \delta_{jk}\delta_k$ for $k \geq 1$. Let $O := M \bar{\otimes} N$ denote the tensor product von Neumann algebra equipped with the tensor product trace. Then $1 \otimes e_{ij}$ for $i, j \geq 1$ are projections of N. If $i, j \geq 1$, there is a compatible contractive operator $(L_1(O), L_\infty(O)) \to (L_1(M), L_\infty(M))$, $y \mapsto y_{ij}$ such that

$$(\tau \otimes \mathrm{tr})((x \otimes e_{ij})y) = \tau(xy_{ij})$$

for every $y \in (L_1 + L_\infty)(O)$ and $x \in (L_1 \cap L_\infty)(M)$. The *matrix coefficients* of $y \in (L_1 + L_\infty)(O)$ is the family $(y_{ij})_{i,j\geq 1}$.

Let P, Q denote the compatible contractive idempotent operators $(L_1(O), L_\infty(O)) \to (L_1(O), L_\infty(O))$ sucht that

$$P(y) = (1 \otimes e_{11})y \qquad Q(y) = y(1 \otimes e_{11})$$

for every $y \in (L_1 + L_\infty)(O)$. Then we have

$$P(x \otimes e_{ij}) = \begin{cases} x \otimes e_{ij} & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$Q(x \otimes e_{ij}) = \begin{cases} x \otimes e_{ij} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

for every $x \in (L_1 + L_\infty)(M)$ and $i, j \geq 1$. Moreover, if $y \in (L_1 \cap L_\infty)(O)$ and $y' \in (L_1 \cap L_\infty)(O)$ then

$$(\tau \otimes \tau_G)(P(y)y') = (\tau \otimes \tau_G)(yQ(y')) = (\tau \otimes \tau_G)(P(y)Q(y')).$$

As a consequence, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$ with Köthe dual $E^\times(O)$, then the pairs $P : E(O) \to E(O)$, $Q : E^\times(O) \to E^\times(O)$ and $Q : E(O) \to E(O)$, $P : E^\times(O) \to E^\times(O)$ are mutually dual operators w.r.t Köthe duality.. In particular $P : E(O) \to E(O)$ and $Q : E(O) \to E(O)$ are weakly continuous w.r.t. Köthe duality.

If $1 \leq p \leq \infty$, the *column space* $L_p(M, \ell_2^c)$ and the *row space* $L_p(M, \ell_2^r)$ are defined as the range of the contractive idempotent operators $L_p(O) \to L_p(O)$ induced by P and Q respectively. They are weakly closed subspaces of $L_p(O)$ w.r.t. Köthe duality.

By definitions, the subfamilies $(L_p(M, \ell_2^c))_{p\in[1,\infty]}$ $(L_p(M, \ell_2^r))_{p\in[1,\infty]}$ are 1-complemented in the compatible family $(L_p(O))_{p\in[1,\infty]}$. Thus, we automatically deduce the following result, showing in particular that the compatible families $(L_p(M, \ell_2^c))_{p\in[1,\infty]}$ and $(L_p(M, \ell_2^r))_{p\in[1,\infty]}$ are real interpolation scales.

Theorem 5.1. *If $0 < \theta < 1$, then*

$$(L_1(M, \ell_2^c), L_\infty(M, \ell_2^c))_{\theta,p} = L_p(M, \ell_2^c)$$

$$(L_1(M, \ell_2^r), L_\infty(M, \ell_2^r))_{\theta,p} = L_p(M, \ell_2^r)$$

with equivalent norms, with constants depending on p only, where $1/p = (1 - \theta)$.

More generally, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, the *column space* $E(M, \ell_2^c)$ and the *row space* $L_p(M, \ell_2^r)$ are defined as the range of the contractive idempotent operators $E(O) \to E(O)$ induced by P and Q respectively. They are weakly closed subspaces of $E(O)$ w.r.t. Köthe duality.

If $E(O)$, $E(M)$ is an exact interpolation pair for $(L_1(O), L_\infty(O))$, $(L_1(M), L_\infty(M))$, then for every $x \in E(M, \ell_2^c)$ (resp. $x \in E(M, \ell_2^r)$), the sequence $(x_{n1})_{n\geq 1}$ (resp. $(x_{1n})_{n\geq 1}$) is contained in $E(M)$, the serie $\sum_{n\geq 1} x_{n1} \otimes e_{n1}$ (resp. $\sum_{n\geq 1} x_{1n} \otimes \lambda_{1n}$) is contained in $E(M, \ell_2)$ and converges weakly to x in $E(O)$ with respect to Köthe duality.

5.1.2 $L_p(\ell_2^{rc})$-spaces and $L_p(\ell_2^{cr})$-spaces

Let G denote the free group with generators $(g_n)_{n\geq 1}$, and let N denote the left von Neumann algebra of G equipped with its canonical trace denoted τ_G. Let $\lambda : G \to N$,

$g \mapsto \lambda_g$ denote the left regular representation of G. Let $O := M\bar{\otimes}B(\ell_2)$ denote the tensor product von Neumann algebra equipped with the tensor product trace. If $g \in G$, there is a compatible contractive operator $(L_1(O), L_\infty(O)) \to (L_1(M), L_\infty(M))$, $y \mapsto y_g$ such that

$$(\tau \otimes \mathrm{tr})((x \otimes \lambda_g)y) = \tau(xy_g)$$

for every $y \in (L_1 + L_\infty)(O)$ and $x \in (L_1 \cap L_\infty)(M)$. The *Fourier coefficients* of $y \in (L_1 + L_\infty)(O)$ is the family $(y_g)_{g\in G}$.

There are compatible 2-bounded idempotent operators $P, Q : (L_1(O), L_\infty(O)) \to (L_1(O), L_\infty(O))$ such that

$$P(x \otimes \lambda_g) = \begin{cases} x \otimes \lambda_g & \text{if } g \in \{g_n \ : \ n \geq 1\} \\ 0 & \text{otherwise} \end{cases},$$

$$Q(x \otimes \lambda_g) = \begin{cases} x \otimes \lambda_g & \text{if } g \in \{g_n^{-1} \ : \ n \geq 1\} \\ 0 & \text{otherwise} \end{cases}$$

for every $x \in (L_1 + L_\infty)(M)$ and $g \in G$. Moreover, if $y \in (L_1 \cap L_\infty)(O)$ and $y' \in (L_1 \cap L_\infty)(O)$ then

$$(\tau \otimes \mathrm{tr})(P(y)y') = (\tau \otimes \mathrm{tr})(yQ(y')) = (\tau \otimes \mathrm{tr})(P(y)Q(y')).$$

As a consequence, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$ with Köthe dual $E^\times(O)$, then the pairs $P : E(O) \to E(O)$, $Q : E^\times(O) \to E^\times(O)$ and $Q : E(O) \to E(O)$, $P : E^\times(O) \to E^\times(O)$ are mutually dual operators w.r.t Köthe duality.. In particular $P : E(O) \to E(O)$ and $Q : E(O) \to E(O)$ are weakly continuous w.r.t. Köthe duality.

If $1 \leq p \leq \infty$, the *mixed spaces* $L_p(M, \ell_2^{cr})$ and $L_p(M, \ell_2^{rc})$ are defined as the range of the 2-bounded idempotent operators $L_p(O) \to L_p(O)$ induced by P and Q respectively. They are weakly closed subspace of $L_p(O)$ with respect to Köthe duality.

By definition, the subfamilies $(L_p(M, \ell_2^{cr}))_{p\in[1,\infty]}$ $(L_p(M, \ell_2^{rc}))_{p\in[1,\infty]}$ are 2-complemented in the compatible family $(L_p(O))_{p\in[1,\infty]}$ by their very definition. Thus, we automatically deduce the following result, showing in particular that the compatible families $(L_p(M, \ell_2^{cr}))_{p\in[1,\infty]}$ and $(L_p(M, \ell_2^{rc}))_{p\in[1,\infty]}$ are real interpolation scales.

Theorem 5.2. *If $0 < \theta < 1$, then*

$$(L_1(M, \ell_2^{cr}), L_\infty(M, \ell_2^{cr}))_{\theta,p} = L_p(M, \ell_2^{cr})$$

$$(L_1(M, \ell_2^{rc}), L_\infty(M, \ell_2^{rc}))_{\theta,p} = L_p(M, \ell_2^{rc})$$

with equivalent norms, with constants depending on p only, where $1/p = (1 - \theta)$.

As before, more generally, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, the *mixed spaces* $E(M, \ell_2^{cr})$ and $E(M, \ell_2^{rc})$ are defined as the range of the 2-bounded idempotent operators $E(O) \to E(O)$ induced by P and Q respectively. They are weakly closed subspaces of $E(O)$ w.r.t. Köthe duality.

If $E(O)$, $E(M)$ is an exact interpolation pair for $(L_1(O), L_\infty(O))$, $(L_1(M), L_\infty(M))$, then for every $x \in E(M, \ell_2^{cr})$ (resp. $x \in E(M, \ell_2^{rc})$), the sequence $(x_{g_n})_{n\geq 1}$ (resp. $(x_{g_n^{-1}})_{n\geq 1}$) is contained in $E(M)$, the serie $\sum_{n\geq 1} x_{g_n} \otimes \lambda_{g_n}$ (resp. $\sum_{n\geq 1} x_{g_n^{-1}} \otimes \lambda_{g_n^{-1}}$) is contained in $E(M, \ell_2)$ and converges weakly to x in $E(O)$ with respect to Köthe duality.

5.1.3 $L_p(\ell_2)$-spaces

In this paragraph we introduce a framework that encompasses the study of the four families of $L_p(\ell_2)$-spaces introduced above. Let M be a von Neumann algebra equipped with a (n.s.f.) trace τ. Let N be an auxiliary von Neumann algebra equipped with a (n.s.f.) trace σ, and let $O := M\bar{\otimes}N$ denote the tensor product von Neumann algebra equipped with the tensor product trace $\tau \otimes \sigma$. Let $P : (L_1(O), L_\infty(O)) \to (L_1(O), L_\infty(O))$ be a compatible 2-bounded idempotent operator. If $E(O)$ is an interpolation space for $(L_1(O), L_\infty(O))$, assume that the 2-bounded idempotent operator $E(O) \to E(O)$ induced by P is weakly continuous w.r.t. Köthe duality and let $E(M, \ell_2)$ denote its range, so that $E(M, \ell_2)$ is a weakly closed subspace of $E(O)$ w.r.t. Köthe duality. Also assume that there is a subset X of $L_1(N) \cap L_\infty(N)$ that is orthonormal in $L_2(N)$, whose linear span is a weak*-dense $*$-subalgebra A of N, and a sequence $(\xi_n)_{n\geq 1}$ of X that generates A as a $*$-algebra such that

$$P(x \otimes \xi) = \begin{cases} x \otimes \xi & \text{if } \xi \in \{\xi_n \ : \ n \geq 1\} \\ 0 & \text{otherwise} \end{cases} .$$

for every $x \in (L_1 + L_\infty)(M)$ and $\xi \in X$. If $\xi \in X$, there is a compatible contractive operator $(L_1(O), L_\infty(O)) \to (L_1(M), L_\infty(M))$, $y \mapsto y_\xi$ such that

$$(\tau \otimes \sigma)((x \otimes \xi)y) = \tau(xy_\xi)$$

for every $y \in (L_1 + L_\infty)(O)$ and $x \in (L_1 \cap L_\infty)(M)$. For $x \in L_1(M, \ell_2) + L_\infty(M, \ell_2)$ and $n \geq 1$, we set $x_n := x_{\xi_n}$. The sequence $(x_n)_{n\geq 1}$ is the *coefficients sequence* of x.

If $E(O)$, $E(M)$ is an exact interpolation pair for $(L_1(O), L_\infty(O))$, $(L_1(M), L_\infty(M))$, and if $x \in E(M, \ell_2)$, then the sequence $(x_n)_{n\geq 1}$ is contained in $E(M)$, the serie $\sum_{n\geq 1} x_{\xi_n} \otimes \xi_n$ is contained in $E(M, \ell_2)$ and we will assume that it converges weakly to x in $E(O)$ with respect to Köthe duality.

5.2 Results

In this final paragraph, we place ourselves within the framework introduced above.

5.2.1 Preliminaries

Let $(M_n)_{n\geq 1}$ be a filtration on M with associated conditional expectations denoted $(E_n)_{n\geq 1}$. For every $n \geq 1$, let N_n be the von Neumann subalgebra of N generated by $(\xi_k)_{k\leq n}$. As the sequence $(\xi_n)_{n\geq 1}$ belongs to $L_1(N) \cap L_\infty(N)$, there is a (trace-preserving normal faithful) conditional expectation of N onto N_n. As the sequence $(\xi_n)_{n\geq 1}$ clearly generates N as a von Neumann algebra, it follows that $(N_n)_{n\geq 1}$ is a filtration on N. Let $(F_n)_{n\geq 1}$ denote the associated conditional expectations. As the sequence $(\xi_n)_{n\geq 1}$ is orthonormal in $L_2(N)$, for every $n, k \geq 1$ we have

$$F_n(\xi_k) = \begin{cases} \xi_k & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}.$$

Let $(O_n^-)_{n\geq 1}$, $(O_n^+)_{n\geq 1}$ be the two filtrations on O such that, if $n \geq 1$,

$$O_{2n-1}^- := M_n \bar{\otimes} N_n, \qquad O_{2n}^- := M_{n+1} \bar{\otimes} N_n,$$

and

$$O_{2n-1}^+ := M_n \bar{\otimes} N_{n+1}, \qquad O_{2n}^+ := M_{n+1} \bar{\otimes} N_{n+1}.$$

Let $(E_n^-)_{n\geq 1}, (E_n^+)_{n\geq 1}$ denote their associated conditional expectations and $(D_n^-)_{n\geq 1}$, $(D_n^+)_{n\geq 1}$ their associated increment projections. Thus, for $n \geq 1$ we have

$$E_{2n-1}^- := E_n \otimes F_n, \qquad E_{2n}^- := E_{n+1} \otimes F_n,$$
$$E_{2n-1}^+ := E_n \otimes F_{n+1}, \qquad E_{2n}^+ := E_{n+1} \otimes F_{n+1}.$$

Lemma 5.3. *The operator $P : (L_1 + L_\infty)(O) \to (L_1 + L_\infty)(O)$ commutes with the conditional expectations $(E_n^\pm)_{n\geq 1}$.*

Proof. Let $n \geq 1$. As the linear span of X is a weak*-dense $*$-subalgebra of N, and because P is weakly continuous with respecto to Köthe duality, it suffices to check that P commutes with $E_{2n}^\pm$ and $E_{2n-1}^\pm$ on elements of $(L_1 + L_\infty)(O)$ of the form $x \otimes \xi$ with $x \in (L_1 + L_\infty)(M)$ and $\xi \in X$. For instance, we have

$$\begin{aligned} E_{2n-1}^- P(x \otimes \xi) &= \begin{cases} E_{2n-1}^-(x \otimes \xi) & \text{if } \xi \in \{\xi_k \ : \ k \geq 1\} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} E_n(x) \otimes F_n(\xi) & \text{if } \xi \in \{\xi_k \ : \ k \geq 1\} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} E_n(x) \otimes \xi & \text{if } \xi \in \{\xi_k \ : \ k \leq n\} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

and

$$\begin{aligned} PE^{-}_{2n-1}(x\otimes\xi) &= P(E_n(x)\otimes F_n(\xi)) \\ &= \begin{cases} E_n(\xi)\otimes F_n(\xi) & \text{if } F_n(\xi)\in\{\xi_k \ : \ k\geq 1\} \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} E_n(\xi)\otimes \xi & \text{if } \xi\in\{\xi_k \ : \ k\leq n\} \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

Thus P commutes with E^{-}_{2n-1}. The other cases are checked analogously. $\square$

If $E(O)$ is an interpolation space for $(L_1(O), L_\infty(O))$, we set

$$E^{\mathrm{ad}}(O) := \Big\{x\in E(O) \ : \ \forall n\geq 1,\ D^{-}_{2n}(x)=0\Big\}.$$

$$E^{\mathrm{mi}}(O) := \Big\{x\in E(O) \ : \ \forall n\geq 1,\ D^{-}_{2n}(x)=D^{+}_{2n-1}(x)=0\Big\}.$$

They are weakly closed subspaces of $E(O)$ w.r.t Köthe duality. In fact, the spaces $E^{\mathrm{ad}}(O)$ and $E^{\mathrm{mi}}(O)$ are particular instances of the spaces $E^{\mathrm{sub}}(O)$ defined in the context of Theorem A and C respectively, which yields the following two results.

Theorem 5.4. *The two following assertions hold.*

1. *If $1\leq p,q\leq\infty$, then the subcouple $(L^{\mathrm{ad}}_p(O), L^{\mathrm{ad}}_q(O))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*
2. *Let $1\leq p,q\leq\infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*

$$E^{\mathrm{ad}}(O) = K_\Phi(L^{\mathrm{ad}}_p(O), L^{\mathrm{ad}}_q(O))$$

   with equivalent norms, with universal constants.

Theorem 5.5. *The two following assertions hold.*

1. *If $1\leq p,q\leq\infty$, then the subcouple $(L^{\mathrm{mi}}_p(O), L^{\mathrm{mi}}_q(O))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*
2. *Let $1\leq p,q\leq\infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*

$$E^{\mathrm{mi}}(O) = K_\Phi(L^{\mathrm{mi}}_p(O), L^{\mathrm{mi}}_q(O))$$

   with equivalent norms, with universal constants.

In addition, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, then by Lemma 5.3, the operator P stabilises the subspaces $E^{\mathrm{ad}}(O)$ and $E^{\mathrm{mi}}(O)$, and thus P induces two 2-bounded idempotent operators $E^{\mathrm{ad}}(O)\to E^{\mathrm{ad}}(O)$ and $E^{\mathrm{mi}}(O)\to E^{\mathrm{mi}}(O)$ whith range $E^{\mathrm{ad}}(O)\cap P(E(O)) = E^{\mathrm{ad}}(O)\cap E(M,\ell_2)$ and $E^{\mathrm{mi}}(O)\cap P(E(O)) = E^{\mathrm{mi}}(O)\cap E(M,\ell_2)$ respectively.

5.2.2 $L_p^{\mathrm{ad}}(\ell_2)$-spaces

Recall that a sequence $(x_n)_{n\geq 1}$ of $(L_1+L_\infty)(M)$ is said to be adapted if $E_n(x_n) = x_n$ for every $n \geq 1$. For $1 \leq p \leq \infty$, we set

$$L_p^{\mathrm{ad}}(M,\ell_2) := \Big\{x \in L_p(M,\ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is adapted}\Big\}.$$

More generally, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, we set

$$E^{\mathrm{ad}}(M,\ell_2) := \Big\{x \in E(M,\ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is adapted}\Big\}.$$

It is a closed subspace of $E(O)$ w.r.t. Köthe duality.

Lemma 5.6. *Let $x \in L_1(M,\ell_2) + L_\infty(M,\ell_2)$. Then the sequence $(x_n)_{n\geq 1}$ is adapted if and only if $D_{2n}^-(x) = 0$ for every $n \geq 1$. As a consequence, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$ then $E^{\mathrm{ad}}(O) \cap E(M,\ell_2) = E^{\mathrm{ad}}(M,\ell_2)$.*

Proof. As the serie $\sum_{k\geq 1} x_k \otimes \xi_k$ converges weakly to x in $(L_1+L_\infty)(O)$ w.r.t. Köthe duality, for $n \geq 1$ we have

$$\begin{aligned} D_{2n}^-(x) &= \sum_{k\geq 1} D_{2n}^-(x_k \otimes \xi_k) \\ &= \sum_{k\geq 1} (E_{2n}^- - E_{2n-1}^-)(x_k \otimes \xi_k) \\ &= \sum_{k\geq 1} (E_{n+1}\otimes F_n - E_n \otimes F_n)(x_k \otimes \xi_k) \\ &= \sum_{k=1}^{n} (E_{n+1} - E_n)(x_k) \otimes \xi_k. \end{aligned}$$

As a result, $D_{2n}(Tx) = 0$ for every $n \geq 1$ if and only if $(E_{n+1} - E_n)(x_k) = 0$ for every $1 \leq k \leq n$. The conclusion follows since for every $k \geq 1$ we have

$$x_k = E_k(x_k) + \sum_{n\geq k} (E_{n+1} - E_n)(x_k)$$

where the sum converges weakly in $(L_1+L_\infty)(M)$ w.r.t. Köthe duality. □

From the above lemma and the last remark in the preliminaries, we directly deduce that if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, then P induces a 2-bounded idempotent operator $P : E^{\mathrm{ad}}(O) \to E^{\mathrm{ad}}(O)$ whose range is $E^{\mathrm{ad}}(M,\ell_2)$. From Theorem 5.4, we automatically deduce the following result.

Theorem 5.7. *The two following assertions hold.*

1. *If $1 \leq p, q \leq \infty$, then $(L_p^{\mathrm{ad}}(M,\ell_2), L_q^{\mathrm{ad}}(M,\ell_2))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*
2. *Let $1 \leq p, q \leq \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*
$$E^{\mathrm{ad}}(M,\ell_2) = K_\Phi(L_p^{\mathrm{ad}}(M,\ell_2), L_q^{\mathrm{ad}}(M,\ell_2))$$
with equivalent norms, with universal constants.

5.2.3 $L_p^{\mathrm{mi}}(\ell_2)$-spaces

Recall that a sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is said to be a martingale increment if it is adapted and if $E_{n-1}(x_n) = 0$ for every $n \geq 2$. For $1 \leq p \leq \infty$, we set
$$L_p^{\mathrm{mi}}(M,\ell_2) := \Big\{x \in L_p(M,\ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment with } x_1 = 0\Big\}.$$
More generally, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, we set
$$E^{\mathrm{mi}}(M,\ell_2) := \Big\{x \in E(M,\ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment with } x_1 = 0\Big\}.$$
It is a closed subspace of $E(O)$ w.r.t. Köthe duality.

Lemma 5.8. *Let $x \in L_1(M,\ell_2) + L_\infty(M,\ell_2)$. Then the sequence $(x_n)_{n\geq 1}$ is a martingale increment with $x_1 = 0$ if and only if $D_{2n}^-(x) = 0$ and $D_{2n-1}^+(x) = 0$ for every $n \geq 1$. As a consequence, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$ then $E^{\mathrm{mi}}(O) \cap E(M,\ell_2) = E^{\mathrm{mi}}(M,\ell_2)$.*

Proof. We already know that the sequence $(x_n)_{n\geq 1}$ is adapted if and only if $D_{2n}^-(x) = 0$ for every $n \geq 1$. Thus it suffices to show that $D_{2n-1}^+(x) = 0$ for every $n \geq 1$ if and only if $E_1(x_1) = 0$ and $E_{n-1}(x_n) = 0$ for every $n \geq 2$. As the serie $\sum_{k\geq 1} x_k \otimes \xi_k$ converges weakly to x in $(L_1 + L_\infty)(O)$ w.r.t. Köthe duality, for $n \geq 2$ we have
$$\begin{aligned}
D_{2n-1}^+(x) &= \sum_{k\geq 1} D_{2n-1}^+(x_k \otimes \xi_k) \\
&= \sum_{k\geq 1} (E_{2n-1}^+ - E_{2(n-1)}^+)(x_k \otimes \xi_k) \\
&= \sum_{k\geq 1} (E_n \otimes F_{n+1} - E_n \otimes F_n)(x_k \otimes \xi_k) \\
&= \sum_{k\geq 1} E_n(x_k) \otimes (F_{n+1} - F_n)(\xi_k) \\
&= E_n(x_{n+1}) \otimes \xi_{n+1}
\end{aligned}$$

and also

$$\begin{aligned}D_1^+(x) = E_1^+(x) &= \sum_{k\geq 1} E_1^+(x_k \otimes \xi_k) \\ &= \sum_{k\geq 1} (E_1 \otimes F_2)(x_k \otimes \xi_k) \\ &= E_1(x_1) \otimes \xi_1 + E_1(x_2) \otimes \xi_2.\end{aligned}$$

The conclusion follows since the sequence $(\xi_n)_{n\geq 1}$ is linearly free. □

From the above lemma and the last remark in the preliminaries, we directly deduce that if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, then P induces a 2-bounded idempotent operator $P : E^{\mathrm{mi}}(O) \to E^{\mathrm{mi}}(O)$ whose range is $E^{\mathrm{mi}}(M, \ell_2)$. From Theorem 5.5, we automatically deduce the following result.

Theorem 5.9. *The two following assertions hold.*

1. *If $1 \leq p, q \leq \infty$, then $(L_p^{\mathrm{mi}}(M, \ell_2), L_q^{\mathrm{mi}}(M, \ell_2))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*
2. *Let $1 \leq p, q \leq \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*
$$E^{\mathrm{mi}}(M, \ell_2) = K_\Phi(L_p^{\mathrm{mi}}(M, \ell_2), L_q^{\mathrm{mi}}(M, \ell_2))$$
with equivalent norms, with universal constants.

5.2.4 $L_p^{\mathsf{hardy}}(\ell_2)$-spaces

For $1 \leq p \leq \infty$, we set

$$L_p^{\mathrm{hardy}}(M, \ell_2) := \Big\{x \in L_p(M, \ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment}\Big\}.$$

More generally, if $E(O)$ is an exact interpolation space for $(L_1(O), L_\infty(O))$, we set

$$E^{\mathrm{hardy}}(M, \ell_2) := \Big\{x \in E(M, \ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment}\Big\}.$$

It is a closed subspace of $E(O)$ w.r.t. Köthe duality. From now, on we will assume that there are compatible 2-bounded idempotent operators $A, B : (L_1(M, \ell_2), L_\infty(M, \ell_2) \to (L_1(M, \ell_2), L_\infty(M, \ell_2))$ satisfying the following properties.

- If $x \in L_1(M, \ell_2) + L_\infty(M, \ell_2)$, then $(Ax)_n = 1_{\{n\geq 2\}} x_n$ for every $n \geq 1$.

- If $x \in L_1(M, \ell_2) + L_\infty(M, \ell_2)$, then $(Bx)_n = 1_{\{n=1\}} E_1(x_1)$ for every $n \geq 1$.

It can be easily shown that the existence of such operators is guaranted in the context of column-row-mixed spaces, so that all the present abstract setting still encompasses the interesting examples.

Theorem 5.10. *The two following assertions hold.*

1. *If $1 \leq p, q \leq \infty$, then $(L_p^{\mathrm{hardy}}(M, \ell_2), L_q^{\mathrm{hardy}}(M, \ell_2))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant.*
2. *Let $1 \leq p, q \leq \infty$ and let Φ be a K-parameter space such that the exact interpolation space $E(O) := K_\Phi(L_p(O), L_q(O))$ has order continuous norm. Then*
$$E^{\mathrm{hardy}}(M, \ell_2) = K_\Phi(L_p^{\mathrm{hardy}}(M, \ell_2), L_q^{\mathrm{hardy}}(M, \ell_2))$$
with equivalent norms, with universal constants.

Proof. Let $x \in L_p^{\mathrm{hardy}}(M, \ell_2) + L_q^{\mathrm{hardy}}(M, \ell_2)$, $y \in L_p(O)$, $z \in L_q(O)$ such that
$$x = y + z.$$
Then
$$x = Px = Py + Pz$$
and
$$Ax = APy + APz.$$
From the properties of the operator A, it is clear that $A(x) \in L_p^{\mathrm{mi}}(M, \ell_2) + L_q^{\mathrm{mi}}(M, \ell_2)$. As the subcouple $(L_p^{\mathrm{mi}}(M, \ell_2), L_q^{\mathrm{mi}}(M, \ell_2))$ is K-complemented in the compatible couple $(L_p(O), L_q(O))$ with a universal constant, we deduce that we have a decomposition
$$Ax = y' + z',$$
where $y' \in L_p^{\mathrm{mi}}(M, \ell_2)$, $z' \in L_q^{\mathrm{mi}}(M, \ell_2)$, with
$$\|y'\|_{L_p(O)} \leq C\|Ay\|_{L_p(O)} \leq 2C\|y\|_{L_p(O)}, \qquad \|z'\|_{L_q(O)} \leq C\|Az\|_{L_q(O)} \leq 2C\|z\|_{L_q(O)},$$
where $C > 0$ is a universal constant. Finally, we set
$$y'' := y' + BPy \in L_p(M, \ell_2), \qquad z'' := z' + BPz \in L_q(M, \ell_2).$$
From the properties of the operator B, it is clear that we have $y'' \in L_p^{\mathrm{hardy}}(M, \ell_2)$ and $z'' \in L_q^{\mathrm{hardy}}(M, \ell_2)$. Moreover, by the triangle inequality, we have
$$\|y''\|_{L_p(O)} \leq \|y'\|_{L_p(O)} + \|BP(y)\|_{L_p(O)} \leq (2C + 4)\|y\|_{L_p(O)}$$

and

$$\|z''\|_{L_q(O)} \leq \|z'\|_{L_q(O)} + \|BP(z)\|_{L_q(O)} \leq (2C+4)\|z\|_{L_q(O)}$$

The proof of the first part of the theorem is thus proved. For the second part, we know that we have an inclusion operator

$$K_\Phi(L_p^{\mathrm{hardy}}(M,\ell_2), L_q^{\mathrm{hardy}}(M,\ell_2)) \to K_\Phi(L_p(O), L_q(O)) = E(O)$$

which is an embedding of normed spaces, with universal constants, and with range

$$(L_p^{\mathrm{hardy}}(M,\ell_2) + L_q^{\mathrm{hardy}}(M,\ell_2)) \cap E(O).$$

Thus it suffices to show that $(L_p^{\mathrm{hardy}}(M,\ell_2) + L_q^{\mathrm{hardy}}(M,\ell_2)) \cap E(O)$ is a norm-dense subspace of $E^{\mathrm{hardy}}(M,\ell_2)$. It is clear that it is indeed a subspace of $E^{\mathrm{hardy}}(M,\ell_2)$. Finally, let $x \in E^{\mathrm{hardy}}(M,\ell_2)$, and set $E(M) := K_\Phi(L_p(M), L_q(M))$. As $E(O)$, $E(M)$ is an exact interpolation pair for $(L_1(O), L_\infty(O)$, $(L_1(M), L_\infty(M))$, we know that the sequence $(x_n)_{n\geq 1}$ is contained in $E(M)$, that the serie $\sum_{n\geq 1} x_{\xi_n} \otimes \xi_n$ is contained in $E(M,\ell_2)$ and converges weakly to x in $E(O)$ with respect to Köthe duality, so it converges in norm x in $E(O)$ because $E(O)$ has order continuous norm. Thus, it suffices to show that, for $n \geq 1$ fixed, we have

$$\sum_{k=1}^{n} x_k \otimes \xi_k \in L_p^{\mathrm{hardy}}(M,\ell_2) + L_q^{\mathrm{hardy}}(M,\ell_2).$$

As the sequence $(x_k)_{k\geq n}$ is a martingale increment, for $k \leq n$ we have $x_k = D_k(\sum_{i=1}^n x_i)$. Moreover, as $E(M)$ is intermediate for $(L_p(M), L_q(M))$, we have a decomposition $\sum_{k=1}^n x_k = y_n + z_n$ with $y_n \in L_p(M)$, $z_n \in L_q(M)$. Thus, we have

$$\sum_{k=1}^{n} x_k \otimes \xi_k = \sum_{k=1}^{n} D_k(y_n) \otimes \xi_k + \sum_{k=1}^{n} D_k(z_n) \otimes \xi_k.$$

As clearly $\sum_{k=1}^n D_k(y_n) \otimes \xi_k \in L_p^{\mathrm{hardy}}(M,\ell_2)$ and $\sum_{k=1}^n D_k(z_n) \otimes \xi_k \in L_q^{\mathrm{hardy}}(M,\ell_2)$, the proof is complete. $\square$

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