A note on Pisier's method for complex interpolation

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In his approach to the interpolation of Hardy spaces on the torus, Pisier described a general method that allowed him to derive complex interpolation properties from real interpolation properties. The purpose of this paper is to formulate this method in a more general setting. We obtain a general result that allows us to derive in a systematic way that a compatible family is well-behaved under complex interpolation if it is well-behaved under real interpolation. As an application, we recover Musat's theorem about complex interpolation of martingale Hardy spaces.

Contents

1 Introduction

Interpolation of Hardy type spaces have been widely investigated since Jones established in [5] that the family of classical Hardy spaces on the torus forms a complex interpolation scale. Using the well-known connections between real and complex interpolation, this implies that this family is also a real interpolation scale. In [7], Pisier first established that the family of classical Hardy spaces on the torus forms a real interpolation scale, and then by using an amplification trick deduce the analogous result for the complex method. In this paper, we formulate this method in a more general setting.

To better explain our considerations, we now introduce the mathematical setting of the paper. We refer to the body of the paper for unexplained notations in the following. Let M be a tracial von Neumann algebra equipped with a filtration and let $(D_n)_{n\geq 1}$ denote the associated increment projections. For a fixed set of positive integers I, let $L_p^{\mathrm{sub}}(M)$ denote the closed subspace of $L_p(M)$ of elements $x \in L_p(M)$ such that $D_n(x) = 0$ for every $n \notin I$. The properties of theses spaces under real interpolation is elucidated in [6]. Using Pisier's method to pass from the real method to the complex method, we establish that, for $1 \leq p_0, p_1 < \infty$ and $0 < \theta < 1$, we have

$$(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))_\theta = L_{p_\theta}^{\mathrm{sub}}(M) \tag{1.1}$$

with equivalent norms, with constants depending on p_0, p_1, θ only, and where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. We also establish an analogous result when M is equipped with two filtrations.

As an application, we obtain a new proof of Musat's theorem about interpolation between noncommutative L_qMO-spaces and L_p-spaces. Let M be a tracial von Neumann algebra equipped with a filtration. For $1 \leq p \leq \infty$, let $L_p(M, \ell_2^r)$ denote the associated row sequence space as introduced by Pisier and Xu in [8]. For $1 \leq p < \infty$, let $H_p^r(M)$ denote the closed subspace of $L_p(M, \ell_2^r)$ of martingale increment sequences. For $1 < q \leq \infty$, let $L_q^r MO(M)$ denote the dual of $H_p(M)$, where $1 \leq p < \infty$ is such that $1/p + 1/q = 1$, which can be identified as a subspace of $L_2(M)$. If $q = \infty$ then $L_\infty^r MO(M) = BMO^r(M)$ is the noncommutative BMO-space over M. The main result of Musat's paper asserts that, for $1 \leq p < \infty$ and $0 < \theta < 1$, we have

$$(BMO^r(M), L_p(M))_\theta = L_{p_\theta}^{\mathrm{sub}}(M) \tag{1.2}$$

with equivalent norms, with constants depending on p, θ only, and where $1/p_\theta = \theta/p$. As noted by Musat, (1.2) follows from the fact that, for $1 \leq p < \infty$ and $0 < \theta < 1$, we have

$$(H_1^r(M), H_p^r(M))_\theta = H_{p_\theta}^r(M) \tag{1.3}$$

with equivalent norms, with constants depending on p, θ only, and where $1/p_\theta = \theta/p$. In this paper, we obtain (1.3) as a byproduct of (1.1), providing another proof of Musat's results.

2 Preliminaries

In this first section, we recall some basic facts and classical results on interpolation theory, noncommutative L_p-spaces and noncommutative martingales that will be used in the paper.

2.1 Abstract interpolation theory

The material of this section is taken from [4] and [1].

2.1.1 Compatible couples

A *compatible couple* is a couple (E_0, E_1) of subspaces of a common Hausdorff topological vector space E, such that E_j is equipped with a complete norm that makes the inclusion $E_j \to E$ continuous, for $j \in \{0,1\}$. Then the intersection space $E_0 \cap E_1$ and the sum space E_0+E_1 are canonically equipped with the complete norms $\|\cdot\|_{E_0\cap E_1}$ and $\|\cdot\|_{E_0+E_1}$ defined as follows,

$$\|u\|_{E_0\cap E_1} := \max\Big\{\|u\|_{E_0},\ \|u\|_{E_1}\Big\}, \qquad \text{for } u \in E_0 \cap E_1.$$

$$\|u\|_{E_0+E_1} := \inf\Big\{\|u_0\|_{E_0}+\|u_1\|_{E_1}\ \mid\ u = u_0+u_1,\ u_0 \in E_0, u_1 \in E_1\Big\}, \qquad \text{for } u \in E_0+E_1.$$

An *intermediate space* for a compatible couple (E_0, E_1) is a subspace E_θ of E_0+E_1 that contains $E_0\cap E_1$, and that is equipped with a complete norm that makes the inclusions $E_0\cap E_1 \to E_\theta$ and $E_\theta \to E_0+E_1$ both continuous. If E_{θ_0}, E_{θ_1} are intermediate spaces for a compatible couple (E_0, E_1), then their sum $E_{\theta_0}+E_{\theta_1}$ and their intersection $E_{\theta_0}\cap E_{\theta_1}$ are also intermediate spaces for (E_0, E_1) when equipped with the corresponding sum norm $\|\cdot\|_{E_{\theta_0}+E_{\theta_0}}$ and intersection norm $\|\cdot\|_{E_{\theta_0}\cap E_{\theta_1}}$ as defined above.

2.1.2 Compatible bounded operators

Let (E_0, E_1) and (F_0, F_1) be two compatible couples. A *compatible bounded operator* $(E_0, E_1) \to (F_0, F_1)$ is an operator $T : E_0 + E_1 \to F_0 + F_1$ such that, if $j \in \{0,1\}$, then T that maps E_j into F_j, and $T : E_j \to F_j$ is bounded. In this situation, we set

$$\|T\|_{(E_0,E_1)\to(F_0,F_1)} := \max\Big\{\|T\|_{E_0\to F_0}, \|T\|_{E_1\to F_1}\Big\}.$$

Let $T : (E_0, E_1) \to (F_0, F_1)$ be a compatible bounded operator. Note that T is injective/surjective/bijective if and only if $T : E_j \to F_j$ is, for $j \in \{0,1\}$.

We say that T is an *embedding/quotient* of compatible couples if $T : E_j \to F_j$ is an embedding/quotient of normed spaces for $j \in \{0, 1\}$ (recall that a bounded operator $T : E \to F$ between normed spaces is an embedding/quotient if it is injective/surjective and the induced bounded operator $E/\ker T \to \operatorname{ran} T$ is an isomorphism of normed spaces). We say that T is an *isomorphism* of compatible couples if $T : E_j \to F_j$ is an isomorphism of normed spaces, for $j \in \{0, 1\}$.

We say that T is *contractive* if $\|T\|_{(E_0,E_1)\to(E_0,E_1)} \leq 1$. We say that T is an *isometric embedding/coisometric quotient* of compatible couples if $T : E_j \to F_j$ is an isometric embedding/coisometric quotient of normed spaces for $j \in \{0, 1\}$ (recall that a quotient of normed spaces $T : E \to F$ is coisometric if the induced isomorphism of normed spaces $E/\ker T \to F$ is isometric). We say that T is an *isometric isomorphism* of compatible couples if $T : E_j \to F_j$ is an isometric isomorphism of normed spaces, for $j \in \{0, 1\}$.

Remark 2.1. There is an obvious way to define the category of compatible couples and compatible (contractive) bounded operators. The isomorphisms in this category correspond to the (isometric) isomorphisms of compatibles couples.

An *interpolation space* with constant $C \geq 1$ for a compatible couple (E_0, E_1) is an intermediate space E_θ for (E_0, E_1), such that, if $T : (E_0, E_1) \to (E_0, E_1)$ is a compatible bounded operator, then T maps E_θ into itself and the operator $T : E_\theta \to E_\theta$ is bounded, with $\|T\|_{E_\theta \to E_\theta} \leq C\|T\|_{(E_0,E_1)\to(E_0,E_1)}$. An *exact interpolation space* is an interpolation space with constant $C = 1$. The sum/intersection of (exact) interpolation spaces is again an (exact) interpolation space.

More generally, an *interpolation pair* with constant $C \geq 1$ for a pair of compatible couples (E_0, E_1) and (F_0, F_1) is a pair of intermediate spaces E_θ and F_θ for (E_0, E_1) and (F_0, F_1) respectively, such that, if $T : (E_0, E_1) \to (F_0, F_1)$ is a compatible bounded operator, then T maps E_θ into F_θ and the operator $T : E_\theta \to F_\theta$ is bounded, with $\|T\|_{E_\theta \to F_\theta} \leq C\|T\|_{(E_0,E_1)\to(F_0,F_1)}$. An *exact interpolation space* is an interpolation space with constant $C = 1$.

2.1.3 Interpolation functors

An *interpolation functor* with constant $C \geq 1$ is a map $\mathcal{F}$ that assigns to each compatible couple (E_0, E_1) an intermediate space $\mathcal{F}(E_0, E_1)$, such that, if (E_0, E_1), and (F_0, F_1) is a pair of compatible couples, then $\mathcal{F}(E_0, E_1)$ and $\mathcal{F}(F_0, F_1)$ is an exact interpolation pair with constant C for (E_0, E_1) and (F_0, F_1) (in this situation, if (E_0, E_1) is a compatible couple, then $\mathcal{F}(E_0, E_1)$ is necessarily an interpolation space with constant C for (E_0, E_1)). An exact interpolation functor is an interpolation functor with constant $C = 1$.

Remark 2.2. For instance, the map Σ (resp. Δ) that assings to each compatible couple (E_0, E_1) the sum space $E_0 + E_1$ (resp. the intersection space $E_0 \cap E_1$) is an exact interpolation functor.

Remark 2.3. If $\mathcal{F}$ is an (exact) interpolation functor, then $\mathcal{F}$ defines in an obvious way a functor from the category of compatible couples and compatible (contractive) bounded operators to the category of complete normed spaces and (contractive) bounded operators.

2.1.4 Subcouples

A *subcouple* of a compatible couple (E_0, E_1) is a couple (A_0, A_1) where A_j is a closed subspace of E_j for $j \in \{0, 1\}$. In this situation, the couple (A_0, A_1) inherits a canonical structure of compatible couple, so that the inclusion $A_0 + A_1 \to E_0 + E_1$ becomes an isometric embedding of compatible couples. Thus, if $\mathcal{F}$ is an (exact) interpolation functor, then $\mathcal{F}(A_0, A_1) \subset \mathcal{F}(E_0, E_1)$ continuously (contractively), but the inclusion $\mathcal{F}(A_0, A_1) \to \mathcal{F}(E_0, E_1)$ may not be an embedding of normed spaces.

2.1.5 Complementation

A subcouple (A_0, A_1) of a compatible couple (E_0, E_1) is *(1-)complemented* if there is an compatible (contractive) bounded operator $P : (E_0, E_1) \to (E_0, E_1)$ such that $P : E_j \to E_j$ is idempotent with range A_j, for $j \in \{0, 1\}$. In this situation, if $\mathcal{F}$ is an (exact) interpolation functor, then the inclusion $\mathcal{F}(A_0, A_1) \to \mathcal{F}(E_0, E_1)$ is an (isometric) embedding of normed spaces, and, moreover, we have $\mathcal{F}(A_0, A_1) = \mathcal{F}(E_0, E_1) \cap (A_0 + A_1)$.

2.1.6 Duality

Let (E_0, E_1) be a compatible couple. If E_θ is an intermediate space for (E_0, E_1) such that $E_0 \cap E_1$ is dense in E_θ, then

$$E_\theta^* := \Big\{\phi \in (E_0 \cap E_1)^*, \sup_{u \in E_0 \cap E_1,\ \|u\|_{E_\theta} \leq 1} |\phi(u)| < \infty\Big\}$$

is a subspace of $(E_0 \cap E_1)^*$ and is equipped with the complete norm $\|\cdot\|_{E_\theta^*}$ given by the expression

$$\|\phi\|_{E_\theta^*} = \sup_{u \in E_0 \cap E_1,\ \|u\|_{E_\theta} \leq 1} |\phi(u)|, \qquad \text{for } \phi \in (E_0 \cap E_1)^*.$$

Moreover, it is clear that the inclusion $E_\theta^* \to (E_0 \cap E_1)^*$ is continuous, and there is a canonical isometric isomorphism from E_θ^* to the usual dual of E_θ.

Let (E_0, E_1) be a *regular* compatible couple, i.e. such that $E_0 \cap E_1$ is dense in E_j for $j \in \{0,1\}$. Then, by the above, the couple (E_0^*, E_1^*) is well-defined an inherits a canonical structure of compatible couple. Moreover, we have $E_0^* + E_1^* = (E_0 \cap E_1)^*$ with equal norms and $E_0^* \cap E_1^* = (E_0 + E_1)^*$ with equal norms. As a consequence, if E_θ is an intermediate space for (E_0, E_1) such that $E_0 \cap E_1$ is dense in E_θ, then E_θ^* is an intermediate space for (E_0^*, E_1^*). The following proposition is obvious.

Proposition 2.4. *Let (E_0, E_1) and (F_0, F_1) be two regular compatible couples. If $T : (E_0, E_1) \to (F_0, F_1)$ is a compatible bounded operator, then the duals $T_j^* : F_0^* \to E_0^*$ and $T_j^* : F_1^* \to E_1^*$ of respectively $T_0 : E_0 \to F_0$ and $T_1 : E_1 \to F_1$, are compatible, so that they define a compatible bounded operator denoted $T^* : (F_0^*, F_1^*) \to (E_0^*, E_1^*)$.*

Proof. By definition, if $j \in \{0,1\}$, then $T_j^* : F_j^* \to E_j^*$ is defined by the expression $T_j^*(\phi)(x) = \phi(T_j(x))$ for $\phi \in F_j^*$ and $x \in E_0 \cap E_1$. As $T_0(x) = T_1(x)$ for $x \in E_0 \cap E_1$, the conclusion is straightforward. □

Corollary 2.5. *Let (E_0, E_1) and (F_0, F_1) be two regular compatible couples. If E_θ, F_θ are intermediate spaces for (E_0, E_1), (F_0, F_1), such that $E_0 \cap E_1$ and $F_0 \cap F_1$ are dense in E_θ and F_θ respectively, and such that F_θ^*, E_θ^* is an interpolation pair for (F_0^*, F_1^*), (E_0^*, E_1^*), then E_θ, F_θ is an interpolation pair for (E_0, E_1), (F_0, F_1).*

Proof. Let $T : (E_0, E_1) \to (F_0, F_1)$ be a compatible contractive operator. If $x \in E_0 \cap E_1$ and $\phi \in F_\theta^*$, then by hypothesis we have $T^*(\phi) \in E_\theta^*$ with $\|T^*(\phi)\|_{E_\theta^*} \leq \|\phi\|_{F_\theta^*}$, and thus

$$\|Tu\|_{F_\theta} = \sup_{\|\phi\|_{F_\theta^*} \leq 1} |\phi(Tu)| = \sup_{\|\phi\|_{F_\theta^*} \leq 1} |T^*(\phi)(u)| \leq \sup_{\|\phi\|_{F_\theta^*} \leq 1} \|T^*(\phi)\|_{E_\theta^*} \|u\|_{E_\theta} \leq \|u\|_{E_\theta}.$$

As $E_0 \cap E_1$ is dense in E_θ, the desired conclusion follows. □

In particular, if (E_0, E_1) is a regular compatible couple and if E_θ is an intermediate space for (E_0, E_1) such that $E_0 \cap E_1$ is dense in E_θ, and such that E_θ^* is an exact interpolation space for (E_0^*, E_1^*), then E_θ is an exact interpolation space for (E_0, E_1).

2.1.7 K-functionals

Let (E_0, E_1) be a compatible couple. The K*-functional* of $u \in E_0 + E_1$ is defined for $t > 0$ as

$$K_t(u) = K_t(u, E_0, E_1) := \inf \Big\{ \|u_0\|_{E_0} + t\|u_1\|_{E_1} \ \Big|\ u_0 \in E_0,\ u_1 \in E_1,\ u = u_0 + u_1 \Big\}.$$

For fixed $t > 0$, K_t is an equivalent norm on $E_0 + E_1$. If (E_0, E_1) and (F_0, F_1) are two compatible couples and $T : (E_0, E_1) \to (F_0, F_1)$ a compatible bounded operator, then

$$K_t(Tu, F_0, F_1) \leq \|T\|_{(E_0,E_1)\to(F_0,F_1)} K_t(u, E_0, E_1)$$

for every $u \in E_0 + E_1$ and $t > 0$. In particular, if (A_0, A_1) is a subcouple of a compatible couple (E_0, E_1), then we have $K_t(u, E_0, E_1) \leq K_t(u, A_0, A_1)$ for every $u \in A_0 + A_1$ and $t > 0$.

A *K-method parameter* is a complete normed space $\Phi(t)$ of (equivalent class of) Lebesgue measurable functions with variable $t \in \mathbb{R}_+^*$ such that,

- ▷ if $f(t), g(t) \in \Phi(t)$ with $|g(t)| \leq |f(t)|$ then $\|g(t)\|_{\Phi(t)} \leq \|f(t)\|_{\Phi(t)}$,
- ▷ the function $1 \wedge t$ belongs to $\Phi(t)$.

If $\Phi(t)$ is a K-method parameter and (E_0, E_1) is a compatible couple, then

$$K_\Phi(E_0, E_1) := \Big\{ u \in E_0 + E_1 \ \mid \ K_t(u, E_0, E_1) \in \Phi(t) \Big\}$$

is a subspace of $E_0 + E_1$ and is equipped with the complete norm $\|\cdot\|_{K_\Phi(E_0,E_1)}$ given by the expression

$$\|u\|_{K_\Phi(E_0,E_1)} := \|K_t(u, E_0, E_1)\|_{\Phi(t)}, \qquad \text{for } u \in K_\Phi(E_0, E_1).$$

This construction defines an exact interpolation functor K_Φ called the *K-method* with parameter Φ.

A subcouple (A_0, A_1) of a compatible couple (E_0, E_1) is *K-closed* with constant $C \geq 1$ if $K_t(u, A_1, A_1) \leq C K_t(u, E_0, E_1)$ for every $u \in A_0 + A_1$ and $t > 0$.

Proposition 2.6. *If (A_0, A_1) is a K-closed subcouple of a compatible couple (E_0, E_1), then for every K-method parameter Φ, the inclusion $K_\Phi(A_0, A_1) \to K_\Phi(E_0, E_1)$ is an embedding of normed spaces. Moreover, we have $K_\Phi(A_0, A_1) = (A_0 + A_1) \cap K_\Phi(E_0, E_1)$.*

2.1.8 The real method

Let $0 < \theta < 1$ and $1 \leq p \leq \infty$. Let $\Phi_{\theta,p}(t)$ denote the space of Lebesgue-measurable functions f with variable $t \in \mathbb{R}_+^*$ such that

$$\|f(t)\|_{\Phi_{\theta,p}(t)} := \|t^{-\theta} f(t)\|_{L_p(dt/t)} < \infty$$

Then $\Phi_{\theta,p}(t)$ is a K-parameter space. If (E_0, E_1) be a compatible couple, the *real interpolation space* $(E_0, E_1)_{\theta,p}$ is the K-method interpolation space $\Phi_{\theta,p}(E_0, E_1)$. By convention, we set $(E_0, E_1)_{0,p} := E_0$ and $(E_0, E_1)_{1,p} := E_1$ for every $1 \leq p \leq \infty$.

The following results are taken from [1].

Proposition 2.7. *Let* (E_0, E_1) *be a compatible couple. Then* $E_0 \cap E_1$ *is dense in* $(E_0, E_1)_{\theta,p}$ *for every* $0 < \theta < 1$ *and* $1 \leq p < \infty$*.*

Theorem 2.8 (Duality Theorem)**.** *Let* (E_0, E_1) *be a regular compatible couple. If* $0 < \theta < 1$ *and* $1 \leq p < \infty$ *then* $(E_0, E_1)^*_{\theta,p} = (E_0^*, E_1^*)_{\theta,q}$ *with equivalent norms, with constants depending on* θ *only, and where* $1 < q \leq \infty$ *is such that* $1/p + 1/q = 1$*.*

Theorem 2.9 (Reiteration theorem)**.** *Let* (E_0, E_1) *be a compatible couple. We set*

$$E_{\theta_0} := (E_0, E_1)_{\theta_0,p_0} \qquad \text{and} \qquad E_{\theta_1} := (E_0, E_1)_{\theta_1,p_1},$$

where $0 \leq \theta_0 < \theta_1 \leq 1$ *and* $1 \leq p_0, p_1 \leq \infty$*. Let* $0 < \lambda < 1$ *and* $1 \leq p \leq \infty$*. Then,*

$$(E_{\theta_0}, E_{\theta_1})_{\lambda,p} = (E_0, E_1)_{\theta_\lambda,p}$$

with equivalent norms, where $\theta_\lambda := (1 - \lambda)\theta_0 + \lambda\theta_1$*.*

2.1.9 The complex method

In the sequel, $\mathbb{B} := \{z \in \mathbb{C} \ : \ 0 < \operatorname{Re} z < 1\}$ denote the open unit strip in the complex plane, with boundary $\partial\mathbb{B} = \{it \ : \ t \in \mathbb{R}\} \cup \{1 + it \ : \ t \in \mathbb{R}\}$ and closure $\overline{\mathbb{B}} := \mathbb{B} \cup \partial\mathbb{B}$.

Let (E_0, E_1) be a compatible couple. Let $\mathcal{F}(E_0, E_1)$ denote the space of norm-bounded continuous functions $f : \overline{\mathbb{B}} \to E_0 + E_1$ that are holomorphic on $\mathbb{B}$, such that we have $f(j + it) \in E_j$ for $t \in \mathbb{R}$, $j \in \{0, 1\}$, and such that the function $\mathbb{R} \to E_j$, $t \mapsto f(j + it)$ is continuous and vanishes at infinity, for $j \in \{0, 1\}$. If $0 < \theta < 1$, the *complex interpolation space* $(E_0, E_1)_\theta = (E_0, E_1)_\theta$ is the subspace of $E_0 + E_1$ of elements of the form $f(\theta)$ with $f \in \mathcal{F}(E_0, E_1)$. It is equipped with the complete norm $\| \cdot \|_{(E_0,E_1)_\theta}$ given by the expression

$$\|u\|_{(E_0,E_1)_\theta} := \inf \Big\{ \max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(j + it)\|_{E_j} \ : \ f \in \mathcal{F}(E_0, E_1), \ u = f(\theta) \Big\},$$

for $u \in (E_0, E_1)_\theta$. This construction yields, for fixed $0 < \theta < 1$, an exact interpolation functor. By convention, we set $(E_0, E_1)_0 := E_0$ and $(E_0, E_1)_1 := E_1$.

Proposition 2.10. *Let* (E_0, E_1) *be a compatible couple. Then* $E_0 \cap E_1$ *is dense in* $(E_0, E_1)_\theta$ *for every* $0 < \theta < 1$*.*

The following statement is folklore. It is a consequence of [11][Lemma 2].

Theorem 2.11 (Duality Theorem)**.** *Let* (E_0, E_1) *be a regular compatible couple and* $0 < \theta < 1$*. Then for every* $x \in E_0 \cap E_1$*, we have*

$$C_0 \|x\|_{(E_0,E_1)_\theta} \leq \sup_{\substack{\phi \in (E_0^*, E_1^*)_\theta \\ \|\phi\|_{(E_0^*,E_1^*)_\theta} \leq 1}} |\phi(x)| \leq C_1 \|x\|_{(E_0,E_1)_\theta}$$

where $C_0, C_1 > 0$ *are universal constants.*

The following corollary will be used in the proof of the main result of the paper.

Corollary 2.12. *Let* (E_0, E_1) *be a regular compatible couple and* $0 < \theta < 1$*. Let* E_θ *be intermediate for* (E_0, E_1) *such that* $E_0 \cap E_1$ *is dense in* E_θ*, and such that we have a continuous inclusion* $(E_0^*, E_1^*)_\theta \subset E_\theta^*$ *with constant* $C > 0$*. Then we have a continuous inclusion* $E_\theta \subset (E_0, E_1)_\theta$ *with a constant depending on* C *only.*

Proof. If $x \in E_0 \cap E_1$ then

$$\begin{aligned} \|x\|_{[E_0,E_1]_\theta} &\leq C_0^{-1} \sup_{\substack{\phi \in (E_0^*, E_1^*)_\theta \\ \|\phi\|_{(E_0^*,E_1^*)_\theta} \leq 1}} |\phi(x)| \leq C_0^{-1} \sup_{\substack{\phi \in (E_0^*, E_1^*)_\theta \\ \|\phi\|_{(E_0^*,E_1^*)_\theta} \leq 1}} \|x\|_{E_\theta} \|\phi\|_{E_\theta^*} \\ &\leq (C/C_0) \sup_{\substack{\phi \in (E_0^*, E_1^*)_\theta \\ \|\phi\|_{(E_0^*,E_1^*)_\theta} \leq 1}} \|x\|_{E_\theta} \|\phi\|_{(E_0^*,E_1^*)_\theta} \leq (C/C_0) \|x\|_{E_\theta}. \end{aligned}$$

As $E_0 \cap E_1$ is dense in E_θ, the conclusion follows. □

The following result is taken from [1].

Theorem 2.13 (Reiteration theorem)**.** *Let* (E_0, E_1) *be a compatible couple. We set*

$$E_{\theta_0} := (E_0, E_1)_{\theta_0} \qquad \text{and} \qquad E_{\theta_1} := (E_0, E_1)_{\theta_1},$$

where $0 \leq \theta_0 < \theta_1 \leq 1$*. Let* $0 < \lambda < 1$*. Then, if* (E_0, E_1) *is regular and if* $E_{\theta_0} \cap E_{\theta_1}$ *is dense in* $E_0 \cap E_1$*, we have*

$$(E_{\theta_0}, E_{\theta_1})_\lambda = (E_0, E_1)_{\theta_\lambda}$$

with equal norms, where $\theta_\lambda := (1 - \lambda)\theta_0 + \lambda\theta_1$*.*

2.2 L_p-spaces

The material of this section is taken from [3].

2.2.1 Generalities

Let M be a *tracial von Neumann algebra*, i.e. a von Neumann algebra equipped with a normal semifinite faithful (n.s.f.) trace τ. Let H denote the Hilbert space on which M acts. A closed and densely defined operator x on H with polar decomposition $x = u|x|$ and spectral decomposition $|x| = \int_0^{+\infty} sde_s$ is *affiliated* with M if $u \in M$ and $e_s \in M$ for all $s > 0$. The *distribution function* of x is the right-continuous decreasing function of the variable $s > 0$ denoted λ_x such that

$$\lambda_x(s) = \tau(1 - e_s), \quad \text{for } s > 0.$$

The *singular function* of x is the right-continuous decreasing function of the variable $s > 0$ denote μ_x such that

$$\mu_x(s) := \inf\Big\{t > 0 \ : \ \lambda_x(t) \leq s\Big\}, \quad \text{for } s > 0.$$

A closed and densely defined operator x on H is τ*-measurable* if it is affiliated with M and if its distribution function (or its singular function) takes at leat one finite value. Any element of M is τ-measurable. The set $L_0(M)$ of τ-measurable operators then admits a canonical structure of complete Hausdorff topological $*$-algebra, so that the inclusion $M \to L_0(M)$ is a continuous $*$-morphism with dense range, and τ is canonically extended to the positive part of $L_0(M)$ so that

$$\tau(x) = \int_0^{+\infty} \lambda_x(s)ds = \int_0^{+\infty} \mu_x(s)ds, \quad \text{for } x \in L_0(M)_+.$$

For every $x \in L_0(M)$ and $1 \leq p \leq \infty$ we set

$$\|x\|_p := \begin{cases} \left(\int_0^{+\infty} \lambda_x(s)ps^{p-1}ds\right)^{1/p} = \left(\int_0^{+\infty} \mu_x(s)^p ds\right)^{1/p} & \text{if } p < \infty \\ \inf\{s > 0 \ | \ \lambda_x(s) = 0\} = \sup_{s>0} \mu_x(s) & \text{if } p = \infty \end{cases}.$$

Then, for $1 \leq p \leq \infty$, the *Lebesgue space*

$$L_p(M) := \Big\{x \in L_0(M) \ | \ \|x\|_p < \infty\Big\}$$

is a subspace of $L_0(M)$ and $\|\cdot\|_p$ is a complete norm on $L_0(M)$ that makes the inclusion $L_p(M) \to L_0(M)$ continuous. Moreover, we have $\|x\|_1 = \tau(x)$ for every $x \in L_0(M)_+$ and $\|x\|_\infty = \|x\|_{B(H)}$ so that $L_\infty(M) = M$ with equal norms. In particular, the family $(L_p(M))_{p\in[1,\infty]}$ inherits a canonical structure of compatible family. In the sequel, if $1 \leq p_0, p_1 \leq \infty$ then we use the notations $(L_{p_0} \cap L_{p_1})(M)$ and $(L_{p_0} + L_{p_1})(M)$ as a shorthand for $L_{p_0}(M) \cap L_{p_1}(M)$ and $L_{p_0}(M) + L_{p_1}(M)$ respectively.

Lemma 2.14. *Let $x \in L_0(M)$. Then $x \in (L_1 + L_\infty)(M)$ if and only if for every $t > 0$, we have*

$$\int_0^t \mu_x(s)ds < \infty$$

and in that case we have

$$K_t(x, L_1(M), L_\infty(M)) = \int_0^t \mu_x(s)ds, \qquad \text{for } t > 0.$$

An immediate consequence of this formula we get the following result.

Theorem 2.15. *If $0 < \theta < 1$ then $(L_1(M), L_\infty(M))_{\theta,p} = L_p(M)$ with equivalent norms, with constants depending on θ only, where $1/p = (1-\theta)$.*

By the reiteration theorem for the real method, we automatically deduce the following corollary.

Corollary 2.16. *If $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}(M), L_{p_1}(M))_{\theta,p_\theta} = L_{p_\theta}(M)$$

with equivalent norms, with constants depending on p_0, p_1, θ only, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

For $1 < p < \infty$ and $1 \leq r \leq \infty$ with $r \neq p$, we define the *Lorentz space*

$$L_{p,r}(M) := (L_1(M), L_\infty(M))_{1-1/p,r}$$

By convention, if $1 \leq p \leq \infty$, we set $L_{p,p}(M) := L_p(M)$. For $1 \leq p < \infty$, $1 \leq r \leq \infty$, and $x \in L_0(M)$, we set

$$\|x\|_{p,r} := \begin{cases} \left(p^{1/r} \displaystyle\int_0^{+\infty} (s\lambda_x(s)^{1/p})^r \frac{ds}{s}\right)^{1/r} & \text{if } r < \infty \\ \sup_{s>0} s^{1/p}\mu_x(s) = \sup_{s>0} s\lambda_x(s)^{1/p} & \text{if } r = \infty \end{cases}.$$

Note that $\|\cdot\|_{p,p} = \|\cdot\|_p$ for every $1 < p < \infty$. By convention, we set $\|\cdot\|_{\infty,\infty} := \|\cdot\|_\infty$ and $L_{\infty,\infty}(M) := L_\infty(M)$. The following proposition follows from Holmstedt's formula.

Proposition 2.17. *Let $1 < p < \infty$, $1 \leq r \leq \infty$ and $x \in L_0(M)$. Then $x \in L_{p,r}(M)$ if and only $\|x\|_{p,r} < \infty$, and in that case we have the estimates*

$$\|x\|_{p,r} \leq \|x\|_{L_{p,r}(M)} \leq \frac{1}{1-1/p}\|x\|_{p,r}.$$

As regards complex interpolation, we have the following result.

Theorem 2.18. *If $1 \leq p_0, p_1 < \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}(M), L_{p_1}(M))_\theta = L_{p_\theta}(M)$$

with equal norms, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

2.2.2 Köthe duality

In this paragraph M is a von Neumann algebra equipped with a (n.s.f.) trace τ. Then the trace τ extends to a positive and contractive linear form on $L_1(M)$ still denoted τ.

If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then the *Köthe dual* as defined in [3]

$$E^\times(M) := \Big\{ y \in L_0(M) \ : \ \forall x \in E(M),\ xy \in L_1(M) \Big\}$$

is a subspace of $L_0(M)$ and is equipped with the complete norm $\|\cdot\|_{E^\times(M)}$ given by the expression

$$\|x\|_{E^\times(M)} = \sup_{x \in E(M),\ \|x\|_{E(M)} \leq 1} |\tau(xy)|, \qquad \text{for } x \in E^\times(M).$$

Then $E^\times(M)$ is actually an exact interpolation space for $(L_1(M), L_\infty(M))$.

Proposition 2.19. *If* $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ *with* $1/p_0 + 1/q_0 = 1$ *and* $1/p_1 + 1/q_1 = 1$ *then* $(L_{p_0} + L_{p_1})^\times(M) = (L_{q_0} \cap L_{q_1})(M)$ *and* $(L_{p_0} \cap L_{p_1})^\times(M) = (L_{q_0} + L_{q_1})(M)$ *with equal norms. If* $1 < p, q < \infty$ *and* $1 \leq r, s \leq \infty$ *with* $1/p + 1/q = 1$ *and* $1/r + 1/s = 1$ *then* $L_{p,r}^\times(M) = L_{q,s}(M)$ *with equivalent norms, with constants depending on* p, r *only.*

Remark 2.20. Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. The *Köthe bidual* $E^{\times\times}(M)$ is the Köthe dual of $E^\times(M)$. If $x \in E(M)$ then $x \in E^{\times\times}(M)$ and $\|x\|_{E^{\times\times}(M)} = \|x\|_{E(M)}$, but in general, the inclusion $E(M) \to E^{\times\times}(M)$ may not be surjective. It is surjective if and only if $E(M)$ satisfies *Fatous's lemma*, i.e. if every increasing bounded net $(x_\alpha)_\alpha$ of $E(M)_+$ admits a least upper bound with $\|\sup_\alpha x_\alpha\|_{E(M)} = \sup_\alpha \|x_\alpha\|_{E(M)}$. For example, if $1 \leq p_0, p_1 \leq \infty$ then $(L_{p_0} + L_{p_1})(M)$ and $(L_{p_0} \cap L_{p_1})(M)$ satisfy Fatou's lemma.

Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. Then the bilinear form $E(M) \times E^\times(M) \to \mathbb{C}$, $(x, y) \mapsto \tau(xy)$ defines a canonical duality between $E(M)$ and $E^\times(M)$, called the *Köthe duality* between $E(M)$ and $E^\times(M)$.

Proposition 2.21. *Let* $E(M)$ *be an exact interpolation space for* $(L_1(M), L_\infty(M))$. *Then* $(L_1 \cap L_\infty)(M)$ *is weakly dense in* $E(M)$ *with respect to Köthe duality.*

Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$. If $E^*(M)$ denote the dual of $E(M)$, the Köthe duality between $E(M)$ and $E^\times$ induces a canonical isometric operator $E^\times(M) \to E^*(M)$, but in general it may not be surjective. It is surjective if and only if the norm of $E(M)$ is *order-continuous*, i.e. if for every decreasing net $(x_\alpha)_\alpha$

of $E(M)_+$ such that $\inf_\alpha x_\alpha = 0$ then $\inf_\alpha \|x_\alpha\|_{E(M)} = 0$. Thus, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm, then the weak topology of $E(M)$ w.r.t. Köthe duality actually coincides with its usual weak topology. For example, if $1 \leq p_0, p_1 < \infty$ then $(L_{p_0} + L_{p_1})(M)$ and $(L_{p_0} \cap L_{p_1})(M)$ have order-continuous norm. If $1 < p < \infty$ and $1 \leq r < \infty$, then $L_{p,r}(M)$ has order-continuous norm.

2.2.3 Conditional expectations

Let M be a tracial von Neumann algebra and let N be a von Neumann subalgebra of M such that there is a (trace-preserving normal faithful) conditional expectation E of M onto N. Then N becomes a tracial von Neumann algebra with the restricted trace such that $L_1(N)$ is a subspace of $L_1(M)$ and the inclusion operator $L_1(N) \to L_1(M)$ is isometric. Moreover, the conditional expectation E extends to a contractive compatible operator $(L_1(M), L_\infty(M)) \to (L_1(N), L_\infty(N))$ that restricts to the identity on $L_1(N) + L_\infty(N)$. As a consequence, if $\mathcal{F}$ is an exact interpolation functor then $\mathcal{F}(L_1(N), L_\infty(N))$ is a subspace of $\mathcal{F}(L_1(M), L_\infty(M))$ and the inclusion operator $\mathcal{F}(L_1(N), L_\infty(N)) \to \mathcal{F}(L_1(M), L_\infty(M))$ is isometric. As a consequence, if $\mathcal{F}$ is an exact interpolation functor, then the conditional expectation E induces a canonical contractive operator $\mathcal{F}(L_1(M), L_\infty(N)) \to \mathcal{F}(L_1(N), L_\infty(N))$ which restricts to the identity on $\mathcal{F}(L_1(N), L_\infty(N))$.

2.2.4 Filtrations and martingales

Let M be a tracial von Neumann algebra equipped with a *filtration*, i.e. an increasing sequence $(M_n)_{n\geq 1}$ of von Neumann subalgebras of M whose union $\cup_{n\geq 1} M_n$ is weak*-dense in M and such that there is a trace-preserving normal faithful conditional expectation E_n of M onto M_n for every $n \geq 1$. Then $(E_n)_{n\geq 1}$ is an increasing sequence of commuting projections. For every $n \geq 1$, we set

$$D_n := E_n - E_{n-1}$$

(with the convention $E_0 := 0$). Then $(D_n)_{n\geq 1}$ is a sequence of mutually orthogonal projections that commute with the $(E_n)_{n\geq 1}$. We will refer to them as the *increment projections* associated with the filtration.

A sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is *adapted* if $E_n(x_n) = x_n$ for all $n \geq 1$. A sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is a *martingale* if it is adapted and $E_{n-1}(x_n) = x_{n-1}$ for all $n \geq 2$, and in that case $E_k(x_n) = x_{k\wedge n}$ for every $n, k \geq 1$.

A sequence $(x_n)_{n\geq 1}$ of $(L_1+L_\infty)(M)$ is a *martingale increment* if it is adapted and $E_{n-1}(x_n)=0$ for all $n\geq 2$, and in that case $E_k(x_n)=1_{k\geq n}x_n$ for every $n,k\geq 1$.

If $x\in(L_1+L_\infty)(M)$, the sequence $(E_n(x))_{n\geq 1}$ is a martingale, and the sequence $(D_n(x))_{n\geq 1}$ is a martingale increment. Note that we have $x\in\cup_{n\geq 1}(L_1+L_\infty)(M_n)$ if and only if the sequence $(E_n(x))_{n\geq 1}$ is eventually constant, and also if and only if the sequence $(D_n(x))_{n\geq 1}$ is eventually zero.

Lemma 2.22. *Let $E(M)$ be an exact interpolation space for $(L_1(M),L_\infty(M))$. Then the subspace $\bigcup_{n\geq 1}(L_1\cap L_\infty)(M_n)$ is weakly dense in $E(M)$ with respect to Köthe duality.*

Proof. $\bigcup_{n\geq 1}(L_1\cap L_\infty)(M_n)$ is a $*$-subalgebra of $L_\infty(M)$. Moreover, it is clearly weak*-dense in $L_\infty(M)$ because $\cup_{n\geq 1}L_\infty(M_n)$ is, by definition. Thus $\bigcup_{n\geq 1}(L_1\cap L_\infty)(M_n)$ is norm-dense in $L_1(M)$. As a consequence, it is weakly dense in $(L_1\cap L_\infty)(M)$ and thus also in $E(M)$. □

Theorem 2.23. *Let $E(M)$ be an exact interpolation space for $(L_1(M),L_\infty(M))$ with order continuous norm.*

1. *If $x\in E(M)$, then the sequence $(E_n(x))_{n\geq 1}$ converges (in norm) to x in $E(M)$.*
2. *If $y\in E^\times(M)$, then the sequence $(E_n(y))_{n\geq 1}$ converges weakly to y in $E^\times(M)$ with respect to Köthe duality.*

Proof. Let $x\in E(M)$ and $\epsilon>0$. By the previous lemma, we know that the subspace $\cup_{n\geq 1}(L_1\cap L_\infty)(M_n)$ is weakly dense in $E(M)$, and thus it is norm-dense in $E(M)$ because $E(M)$ has order continuous norm. Thus, there is $y\in E(M)$ and $k\geq 1$ such that $\|x-y\|_{E(M)}<\epsilon$ and $E_k(y)=y$. Then, for all $n\geq k$, we have

$$\begin{aligned}\|E_n(x)-x\|_{E(M)}&=\|E_n(x)+E_n(y)+y-x\|_{E(M)}\\&\leq\|E_n(x-y)\|_{E(M)}+\|x-y\|_{E(M)}\\&\leq 2\|x-y\|_{E(M)}<2\epsilon\end{aligned}$$

which shows that $(E_n(x))_{n\geq 1}$ converges in norm to x. Now, if $y\in E^\times(M)$ then for every $x\in E(M)$ we get

$$\tau(xE_n(y))=\tau(E_n(x)y)\underset{n\to\infty}{\to}\tau(xy)$$

as desired. □

Corollary 2.24. *Let $E(M)$ be an exact interpolation space for $(L_1(M),L_\infty(M))$ with order continuous norm. If $x\in E(M)$ and $y\in E^\times(M)$, then*

$$\tau(xy)=\sum_{n=1}^{+\infty}\tau(D_n(x)D_n(y)).$$

3 Pisier's method

3.1 A weak*-Stein theorem

For the proof of the main result of the paper, we need a version of Stein's interpolation theorem for analytic family of operators that holds in the context of dual spaces equipped with their weak*-topology.

3.1.1 Preliminaries on weak*-holomorphic functions

First, we recall basic facts on weak*-holomorphic functions. Let E^* be a complete normed space equipped with a predual, so that E^* is equipped with a weak*-topology. Let $f : \mathbb{B} \to E^*$ be a function. Then f is holomorphic if and only f is locally norm-bounded and weak*-holomorphic, in the sense that $\phi \circ f$ is holomorphic for every weakly*-continuous linear form ϕ on E^*. This fact is derived easily by adapting the proof of the well-known equality between holomorphic functions and weak-holomorphic functions for functions with values in quasi-complete locally convex spaces.

3.1.2 Statements

Let (F_0^*, F_1^*) be a compatible couple equipped with a predual, i.e. a regular compatible couple (F_0, F_1) together with an isometric isomorphism of compatible couples between (F_0^*, F_1^*) and the dual of (F_0, F_1) as defined in the preliminary section. Let $\mathcal{F}_*(F_0^*, F_1^*)$ denote the space of norm-bounded weak*-continuous functions $f : \overline{\mathbb{B}} \to F_0^* + F_1^*$ (when $F_0^* + F_1^*$ is equipped with the weak*-topology coming from the pairing with $F_0 \cap F_1$), that are (weak*-)holomorphic on $\mathbb{B}$, such that $f(j + it) \in F_j^*$ for $t \in \mathbb{R}$, $j \in \{0, 1\}$, and such that the function $\mathbb{R} \to F_j^*$, $t \mapsto f(j + it)$ is bounded and weak*-continuous F_j^*, for $j \in \{0, 1\}$.

Proposition 3.1. *Let (F_0^*, F_1^*) and (G_0^*, G_1^*) be two compatible couples equipped with preduals. Let $T : (G_0^*, G_1^*) \to (F_0^*, F_1^*)$ be a compatible bounded operator such that the operators $T : G_0^* \to F_0^*$ and $T : G_1^* \to F_1^*$ are weak*-continuous. If $f \in \mathcal{F}_*(G_0^*, G_1^*)$, then the function $z \mapsto T(f(z))$ belongs to $\mathcal{F}_*(F_0^*, F_1^*)$.*

Theorem 3.2 (Weak*-Stein theorem)**.** *Let (E_0, E_1) be a compatible couple and let (F_0^*, F_1^*) be a compatible couple equipped with a predual. Let $(T_z)_{z \in \overline{\mathbb{B}}}$ be a family of operators $E_0 \cap E_1 \to F_0^* + F_1^*$ satisfying the following two conditions.*

1. *For every $u \in E_0 \cap E_1$, the function $\overline{\mathbb{B}} \to F_0^* + F_1^*$, $z \mapsto T_z(u)$ belongs to $\mathcal{F}_*(F_0^*, F_1^*)$.*
2. *For every $u \in E_0 \cap E_1$, $t \in \mathbb{R}$ and $j \in \{0,1\}$, we have $\|T_{j+it}(u)\|_{F_j^*} \leq \|u\|_{E_j^*}$.*

Then, for $u \in E_0 \cap E_1$ and $0 < \theta < 1$ we have $T_\theta(u) \in (F_0^*, F_1^*)_{\theta,\infty}$ with

$$\|T_\theta(u)\|_{(F_0^*,F_1^*)_{\theta,\infty}} \leq \|u\|_{(E_0,E_1)_\theta}.$$

Remark 3.3. It is not known if in the conclusion of the above theorem, the real interpolation space $(F_0^*, F_1^*)_{\theta,\infty}$ can be replaced by the complex interpolation space $(F_0^*, F_1^*)_\theta$. For some results in this direction, see [2].

For the proof, we need a couple of lemma. If (E_0, E_1) is a compatible couple, let $\mathcal{F}_\tau(E_0, E_1)$ denote the set of functions $f : \overline{\mathbb{B}} \to E_0 + E_1$ which can be written as a finite sum $f = \sum_n g_n \otimes u_n$ where $u_n \in E_0 \cap E_1$, and where $g_n : \overline{\mathbb{B}} \to \mathbb{C}$ is a bounded continuous function, holomorphic on $\mathbb{B}$, such that the continuous function $\mathbb{R} \to \mathbb{C}$, $t \mapsto g(j + it)$ vanishes at infinity for $j \in \{0, 1\}$. It is clear that $\mathcal{F}_\tau(E_0, E_1)$ is a subspace of $\mathcal{F}(E_0, E_1)$.

The following lemma appears in [10][Lemma 2.5].

Lemma 3.4. *Let (E_0, E_1) be a compatible couple. Fix $0 < \theta < 1$ and $u \in E_0 \cap E_1$ such that*

$$\|u\|_{(E_0,E_1)_\theta} < 1.$$

Then there is $f \in \mathcal{F}_\tau(E_0, E_1)$ such that $u = f(\theta)$ and

$$\max_{j\in\{0,1\}} \sup_{t\in\mathbb{R}} \|f(j + it)\|_{E_j} < 1.$$

Lemma 3.5. *Let (F_0^*, F_1^*) be a compatible couple equipped with a predual (F_0, F_1). Let $f \in \mathcal{F}_*(F_0^*, F_1^*)$ and $0 < \theta < 1$. Then $f(\theta) \in (F_0^*, F_1^*)_{\theta,\infty}$ with*

$$\|f(\theta)\|_{(F_0^*,F_1^*)_{\theta,\infty}} \leq \max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} \|f(j + is)\|_{F_j^*}.$$

Proof. Fix $t > 0$. If $u \in F_0 \cap F_1$, then by Hadamard's three-lines theorem applied to the function $\overline{\mathbb{B}} \to \mathbb{C}$, $z \mapsto (u, f(z))$, we have

$$|(u, f(\theta))| \leq \sup_{s\in\mathbb{R}} |(u, f(is))|^{1-\theta} \sup_{s\in\mathbb{R}} |(u, f(1 + is))|^\theta.$$

Now, if we denote $C := \max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} \|f(j + it)\|_{F_j^*}$, for $s \in \mathbb{R}$ we have

$$|(u, f(is))| \leq \|u\|_{F_0} \|f(j + is)\|_{F_0^*} \leq C\|u\|_{F_0},$$

$$|(u, f(1 + is))| \leq \|u\|_{F_1} \|f(j + is)\|_{F_1^*} \leq C\|u\|_{F_1} = Ct\|u\|_{t^{-1}F_1}.$$

Thus, by combining the previous estimates, we get

$$|(u, f(\theta))| \leq C\|u\|_{F_0}^{1-\theta} t^\theta \|u\|_{t^{-1}F_1}^\theta \leq Ct^\theta \|u\|_{F_0 \cap t^{-1}F_1}.$$

Finally, we obtain

$$\begin{aligned} K_t(f(\theta), (F_0^*, F_1^*)) &= \|f(\theta)\|_{F_0^* + tF_1^*} \\ &= \sup_{\substack{u \in F_0 \cap tF_1 \\ \|u\|_{F_0 \cap t^{-1}F_1} \leq 1}} |(u, f(\theta))| \\ &\leq \sup_{\substack{u \in F_0 \cap t^{-1}F_1 \\ \|u\|_{F_0 \cap tF_1} \leq 1}} Ct^\theta \|u\|_{F_0 \cap t^{-1}F_1} \\ &\leq Ct^\theta. \end{aligned}$$

This shows indeed that $f(\theta) \in (F_0^*, F_1^*)_{\theta,\infty}$ with $\|f(\theta)\|_{(F_0^*, F_1^*)_{\theta,\infty}} \leq C$. □

Now we are able to complete the proof of our weak*-Stein theorem.

Proof of weak-Stein theorem.* Fix $u \in (E_0, E_1)$ and $0 < \theta < 1$ with $\|u\|_{(E_0,E_1)_\theta} < 1$. By Lemma 3.4, we have $u = f(\theta)$ with $f = \sum_n g_n \otimes u_n \in \mathcal{F}_\tau(E_0, E_1)$ such that

$$\max_{j \in \{0,1\}} \sup_{t \in \mathbb{R}} \|f(j + it)\|_{E_j^*} < 1$$

Let $g : \overline{\mathbb{B}} \to (F_0^*, F_1^*)$ be the function such that $g(z) = T_z(f(z)) = \sum_n g_n(z) T_z(u_n)$ for $z \in \overline{\mathbb{B}}$. By hypothesis, it is clear that g belongs to $\mathcal{F}_*(F_0^*, F_1^*)$. Moreover, if $t \in \mathbb{R}$ then by hypothesis

$$\|g(j + it)\|_{F_j^*} = \|T_{j+it}(f(j + it))\|_{F_j^*} \leq \|f(j + it)\|_{E_j^*} < 1.$$

As we have

$$g(\theta) = T_\theta(f(\theta)) = T_\theta(u),$$

by Lemma 3.5 we deduce that $T_\theta(u) \in (F_0^*, F_1^*)_\theta$ with

$$\|T_\theta(u)\|_{(F_0^*, F_1^*)_\theta} \leq \max_{j \in \{0,1\}} \sup_{t \in \mathbb{R}} \|g(j + it)\|_{F_j^*} < 1.$$

The proof is complete. □

3.2 An amplification map

3.2.1 Preliminaries on amplification maps

In this paragraph we gather basic facts about tensor product of tracial von Neumann algebras. Let M, N be two von Neumann algebras equipped with a (n.s.f.) trace τ, σ respectively.

The following lemma, giving an expression of the distribution function of a tensor product as a convolution product, is folklore.

Proposition 3.6. *If x is affiliated with M and y is affiliated with N, then $x \otimes y$ is affiliated with $M \bar{\otimes} N$ and*

$$\lambda_{x\otimes y}(s) = \int_0^{+\infty} \lambda_x(t)\lambda_y(s/t)dt/t, \qquad \text{for } s > 0.$$

By using Young's inequality for convolution, we easily deduce the following proposition.

Proposition 3.7. *Let $1 \le p \le r \le \infty$. If $x \in L_p(M)$ and $y \in L_{p,r}(N)$, then $x \otimes y \in L_{p,r}(M \bar{\otimes} N)$ with $\|x \otimes y\|_{p,r} \le p^{1/p}\|x\|_p\|y\|_{p,r}$.*

Proposition 3.8. *Let $1 < p \le r \le \infty$ and $x \in L_p(M)$. Then the amplification operator $L_{p,r}(N) \to L_{p,r}(M \bar{\otimes} N)$, $y \mapsto x \otimes y$ is weak*-continuous.*

Proof. Let $1 \le p^*, r^* < \infty$ such that $1/p + 1/p^* = 1$ and $1/r + 1/r^* = 1$. Let T_x denote the operator $(L_1 \cap L_\infty)(M) \otimes (L_1 \cap L_\infty)(N) \to (L_1 \cap L_\infty)(N)$ such that $T_x(u \otimes v) = \tau(xu)v$ for $u \in (L_1 \cap L_\infty)(M)$, $v \in (L_1 \cap L_\infty)(N)$. Then, if $w = \sum_i u_i \otimes v_i$ with $u_i \in (L_1 \cap L_\infty)(M)$, $v_i \in (L_1 \cap L_\infty)$, for $y \in (L_1 \cap L_\infty)(N)$ we have

$$\begin{aligned}\sigma(yT_x(w)) &= \sum_i \tau(xu_i)\sigma(yv_i) = \sum_i (\tau \otimes \sigma)(xu_i \otimes yv_i) \\ &= \sum_i (\tau \otimes \sigma)(x \otimes y)(u_i \otimes v_i) = (\tau \otimes \sigma)((x \otimes y)w)\end{aligned}$$

Thus,

$$\begin{aligned}\|T_x(w)\|_{p^*,r^*} &= \sup_{\|y\|_{p,r}\le 1} |\sigma(yT_x(w))| = \sup_{\|y\|_{p,r}\le 1} |(\tau \otimes \sigma)((x \otimes y)w)| \\ &\le \sup_{\|y\|_{p,r}\le 1} \|x \otimes y\|_{p,r}\|w\|_{p^*,r^*} \le \sup_{\|y\|_{p,r}\le 1} p^{1/p}\|x\|_p\|y\|_{p,r}\|w\|_{p^*,r^*} \\ &\le p^{1/p}\|x\|_p\|w\|_{p^*,r^*}.\end{aligned}$$

Thus T_x extends to a bounded operator $L_{p^*,r^*}(M \bar{\otimes} N) \to L_{p^*,r^*}(N)$ whose dual coincides with the operator $L_{p,r}(N) \to L_{p,r}(M \bar{\otimes} N)$, $y \mapsto x \otimes y$. □

3.2.2 Statements

Let N denote the commutative von Neumann algebra $L_\infty(\mathbb{R})$ equipped with the (n.s.f) trace σ such that

$$\sigma(f) = \int_{-\infty}^{+\infty} f(t)e^t dt, \quad \text{for } f \in N_+.$$

In the sequel $1 < q < \infty$ is fixed and we consider $1 < p < \infty$ such that $1/p + 1/q = 1$. For $z \in \overline{\mathbb{B}}$, we set $q_z := q/(1 - \mathrm{Re}(z)) \in [q, \infty]$. Finally, for $z \in \overline{\mathbb{B}}$, let $f_z \in C(\mathbb{R})$ such that

$$f_z(s) = e^{-s(1-z)/q}, \qquad \text{for } s \in \mathbb{R}.$$

A direct computation shows that, for $z \in \overline{\mathbb{B}}$, the distribution function of f_z as an affiliated operator with N is given by

$$\lambda_{f_z}(s) = s^{-q_z}, \quad \text{for } s > 0.$$

As a consequence, we have $f_z \in L_{q_z,\infty}(N)$ with $\|f_z\|_{q_z,\infty} = 1$. In the sequel, the compatible couple $(L_{q,\infty}(N), L_\infty(N))$ is equipped with its canonical predual coming from its canonical pairing with $(L_{p,1}(N), L_1(N))$ so that the space $\mathcal{F}_*(L_{q,\infty}(N), L_\infty(N))$ is well-defined.

Lemma 3.9. *The function $\overline{\mathbb{B}} \to L_{q,\infty}(N) + L_\infty(N)$, $z \mapsto f_z$ is well-defined and belongs to $\mathcal{F}_*(L_{q,\infty}(N), L_\infty(N))$.*

Proof. If $z \in \overline{\mathbb{B}}$, we know that $L_{q_z,\infty}(N) = (L_{q,\infty}(N), L_\infty(N))_{\mathrm{Re}(z),\infty}$ with equal norms, thus $f_z \in L_{q,\infty}(N) + L_\infty(N)$ with

$$\|f_z\|_{L_{q,\infty}(N)+L_\infty(N)} \leq \|f_z\|_{L_{q_z,\infty}(N)} \leq (1 - q_z^{-1})^{-1}\|f_z\|_{q_z,\infty} \leq (1 - q^{-1})^{-1}.$$

Thus the function $\overline{\mathbb{B}} \to L_{q,\infty}(N) + L_\infty(N)$, $z \mapsto f_z$ is well-defined and norm-bounded. If $t \in \mathbb{R}$ then $f_{it} \in L_{q,\infty}(N)$ with $\|f_{it}\|_{L_{q,\infty}(N)} \leq (1 - q^{-1})^{-1}\|f_{it}\|_{q,\infty} = (1 - q^{-1})^{-1}$ and $f_{1+it} \in L_\infty(N)$ with $\|f_{1+it}\|_{L_\infty(N)} = \|f_{1+it}\|_{\infty,\infty} = 1$. Thus, the functions $\mathbb{R} \to L_{q,\infty}(N)$, $t \mapsto f_{it}$ and $\mathbb{R} \to L_\infty(N)$ are norm-bounded. Now, if $g \in L_1(N)$, then for every $t \in \mathbb{R}$ we have

$$\mu(f_{1+it}g) = \int_{-\infty}^{+\infty} e^{ist/q} g(s) d\mu(s).$$

A direct application of Lebesgue dominated convergence theorem then shows that the function $\mathbb{R} \to \mathbb{C}$, $t \mapsto \mu(f_{1+it}g)$ is continuous. Now, if $g \in L_{p,1}(N)$, then for every $t \in \mathbb{R}$ we have

$$\mu(f_{it}g) = \int_{-\infty}^{+\infty} e^{-s(1-it)/q} g(s) d\mu(s) = \int_{-\infty}^{+\infty} e^{ist/q}|f(s)|g(s) d\mu(s).$$

Again, by the same arguments we see that the function $\mathbb{R} \to \mathbb{C}$, $t \mapsto \mu(f_{it}g)$ is continuous. Thus, the functions $\mathbb{R} \to L_{q,\infty}(N)$, $t \mapsto f_{it}$ and $\mathbb{R} \to L_\infty(N)$, $t \mapsto f_{1+it}$ are weak*-continuous. In a similar way, it is easy to see that the function $\mathbb{B} \to L_{q,\infty}(N)+L_\infty(N)$, $z \mapsto f_z$ is weak*-holomorphic. $\square$

Now, let M be a tracial von Neumann algebra.

Lemma 3.10. *Let $z \in \overline{\mathbb{B}}$ and $x \in L_0(M)$. Then we have*

$$\|x \otimes f_z\|_{q_z,\infty} = q_z^{-1/q_z} \|x\|_{q_z}$$

(with the convention $\infty^0 = 1$).

Proof. If $\mathrm{Re}(z) = 1$ then $|f_z| = 1$ and the conclusion is clear. Thus, we can assume $\mathrm{Re}(z) < 1$. For $s > 0$ we have

$$\begin{aligned}\lambda_{x\otimes f_z}(s) &= \int_0^\infty \lambda_x(t) f_z(s/t)dt/t = \int_0^\infty \lambda_x(t)(s/t)^{-q_z}dt/t \\ &= s^{-q_z}\int_0^\infty \lambda_x(t)t^{q_z-1}dt = s^{-q_z}q_z^{-1}\|x\|_{q_z}^{q_z}\end{aligned}$$

and the conclusion follows. $\square$

For $z \in \overline{\mathbb{B}}$, we consider the operator $F_z : L_{q_z}(M) \to L_{q_z,\infty}(M\bar{\otimes}N)$, $x \mapsto x \otimes f_z$. In the sequel, the compatible couple $(L_{q,\infty}(M\bar{\otimes}N), L_\infty(M\bar{\otimes}N))$ is equipped with its canonical predual coming from its canonical pairing with $(L_{p,1}(M\bar{\otimes}N), L_1(M\bar{\otimes}N))$, so that the space $\mathcal{F}_*(L_{q,\infty}(M\bar{\otimes}N), L_\infty(M\bar{\otimes}N))$ is well-defined.

Theorem 3.11. *The family $(F_z)_{z\in\overline{\mathbb{B}}}$ satifies the following conditions.*

1. *for every $x \in (L_q \cap L_\infty)(M)$, the function $\overline{\mathbb{B}} \to L_{q,\infty}(M\bar{\otimes}N) + L_\infty(M\bar{\otimes}N)$, $z \mapsto F_z(x)$ belongs to $\mathcal{F}_*(L_{q,\infty}(M\bar{\otimes}N), L_\infty(M\bar{\otimes}N))$,*
2. *for every $z \in \overline{\mathbb{B}}$ and $x \in L_{q_z}(M)$ we have*

$$\|x\|_{L_{q_z}(M)} \leq q_z^{1/q_z}\|F_z(x)\|_{L_{q_z,\infty}(M\bar{\otimes}N)} \leq (1-q_z^{-1})^{-1}\|x\|_{L_{q_z}(M)}.$$

Proof. The last point is a direct consequence of Lemma 3.10. For the first point, we fix $x \in (L_q \cap L_\infty)(M)$. As the operators $L_{q,\infty}(M) \to L_{q,\infty}(M\bar{\otimes}N)$, $f \mapsto x \otimes f$ and $L_\infty(N) \to L_\infty(M\bar{\otimes}N)$, $f \mapsto x \otimes f$ are both weak*-continuous. By Lemma 3.9 and Proposition 3.1, we deduce that the function $\overline{\mathbb{B}} \to L_{q,\infty}(M\bar{\otimes}N) + L_\infty(M\bar{\otimes}N)$, $z \mapsto x \otimes f_z$ belongs to $\mathcal{F}_*(L_{q,\infty}(M\bar{\otimes}N), L_\infty(M\bar{\otimes}N))$. $\square$

3.3 Theorem A

3.3.1 The statement

Notations.

If M is a tracial von Neumann algebra equipped with a subspace $(L_1+L_\infty)^{\rm sub}(M)$ of $(L_1+L_\infty)(M)$, for every exact interpolation space $E(M)$ for $(L_1(M),L_\infty(M))$, we set

$$E^{\rm sub}(M) := E(M)\cap (L_1+L_\infty)^{\rm sub}(M)$$

and we denote $E^{\rm ort}(M)$ the orthogonal of $(E^\times)^{\rm sub}(M)$ in $E(M)$ w.r.t. Köthe duality. With this definition, it is clear that if $E(M),E(N)$ are two exact interpolation spaces for $(L_1(M),L_\infty(M))$, then

$$(E\cap F)^{\rm sub}(M) = E(M)\cap F^{\rm sub}(M) = E^{\rm sub}(M)\cap F(M) = E^{\rm sub}(M)\cap F^{\rm sub}(M).$$

Definition 3.12. If M is a tracial von Neumann algebra, a subspace $(L_1+L_\infty)^{\rm sub}(M)$ of $(L_1+L_\infty)(M)$ is *admissilbe* if it satisfies the following assertions.

- if $E(M)$ is any exact interpolation space for $(L_1(M),L_\infty(M))$, then $E^{\rm sub}(M)$ is norm-closed in $E(M)$.
- if $E(M)$ is any exact interpolation space for $(L_1(M),L_\infty(M))$ with order continuous norm, then $(L_1\cap L_\infty)^{\rm sub}(M)$ is norm-dense in $E(M)$ and weak*-dense in $(E^\times)^{\rm sub}(M)$.
- if $E(M)$ is any exact interpolation space for $(L_1(M),L_\infty(M))$ with order continuous norm, then $(L_1\cap L_\infty)(M)\cap E^{\rm ort}(M)$ is norm-dense in $E^{\rm ort}(M)$.
- for every $1<p<\infty$, the orthogonal projection $L_2(M)\to L_2(M)$ onto $L_2^{\rm sub}(M)$ is L_p-bounded with a constant depending on p only.
- for every $1\le p,q<\infty$, the subcouple $(L_p^{\rm sub}(M),L_q^{\rm sub}(M))$ is K-closed in the compatible couple $(L_p(M),L_q(M))$ with a universal constant.

Remark 3.13. If M is a tracial von Neumann algebra equipped with an admissible subspace of $(L_1+L_\infty)(M)$, then, by the second point of the above definition, for every exact interpolation space $E(M)$ for $(L_1(M),L_\infty(M))$ with order continuous norm, we have

$$E^{\rm ort}(M) = \Big\{x\in E(M)\ :\ \tau(xy)=0,\ \forall y\in (L_1\cap L_\infty)^{\rm sub}(M)\Big\}.$$

Proposition 3.14. *Let M be a tracial von Neumann algebra equipped with an admissible subspace of $(L_1+L_\infty)(M)$. Let $1\le p,q<\infty$ and let Φ be a K-parameter* *space such that the exact interpolation space* $E(M) := K_\Phi(L_p(M), L_q(M))$ *has order continuous norm. Then*

$$E^{\mathrm{sub}}(M) = K_\Phi(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$$

with equivalent norms, with universal constants.

Proof. As $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in $(L_p(M), L_q(M))$ with a universal constant, we know that the inclusion operator

$$K_\Phi(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M)) \to K_\Phi(L_p(M), L_q(M)) = E(M)$$

is an embedding of normed spaces, with universal constants, and moreover with range $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$. Thus, it suffices to show that $(L_p^{\mathrm{sub}}(M) + L_q^{\mathrm{sub}}(M)) \cap E(M)$ is a norm-dense subspace of $E^{\mathrm{sub}}(M)$. But it is clear that it is a subspace of $E^{\mathrm{sub}}(M)$ that contains $L_1^{\mathrm{sub}}(M) \cap L_\infty(M)$. $\square$

Proposition 3.15. *Let* M *be a tracial von Neumann algebra equipped with an admissible subspace of* $(L_1 + L_\infty)(M)$*. Let* $1 < p, q < \infty$ *and let* $E(M)$ *be an exact interpolation space for* $(L_p(M), L_q(M))$ *with order continuous norm and with Fatou's property. Then the orthogonal projection* $P : L_2(M) \to L_2(M)$ *onto* $L_2^{\mathrm{sub}}(M)$ *is compatible with a compatible bounded idempotent operator* $P : (E(M), E^\times(M)) \to (E(M), E^\times(M))$ *with constant depending on* p, q *only, such that the bounded idempotent operators* $P : E(M) \to E(M)$ *and* $P : E^\times(M) \to E^\times(M)$ *have range* $E^{\mathrm{sub}}(M)$ *and* $(E^\times)^{\mathrm{sub}}(M)$ *respectively, and have kernel* $E^{\mathrm{ort}}(M)$ *and* $(E^\times)^{\mathrm{ort}}(M)$ *respectively. Moreover,* $P : E^\times(M) \to E^\times(M)$ *is the dual of* $P : E(M) \to E(M)$ *w.r.t. Köthe duality.*

Proof. As $P : L_2(M) \to L_2(M)$ is L_p-bounded and L_q-bounded, we deduce that P is compatible with a compatible bounded idempotent operator $P : (L_p(M), L_q(M)) \to (L_p(M), L_q(M))$. As $E(M)$ is an exact interpolation space for $(L_p(M), L_q(M))$, we deduce that P induces a bounded idempotent operator $P : E(M) \to E(M)$. The range of $P : E(M) \to E(M)$ must coincides with the closure of $P(L_2(M)) \cap E(M) = L_2^{\mathrm{sub}}(M) \cap E(M)$ in $E(M)$. As $L_2^{\mathrm{sub}}(M) \cap E(M)$ contains $(L_1 \cap L_\infty)^{\mathrm{sub}}(M)$, we deduce that the range of $P : E(M) \to E(M)$ is $E^{\mathrm{sub}}(M)$. For the same reasons, the kernel of $P : E(M) \to E(M)$ is $E^{\mathrm{ort}}(M)$. Thus, the dual of $P : E(M) \to E(M)$ w.r.t Köthe duality is a bounded idempotent operator $P^\times : E^\times(M) \to E^\times(M)$ with range the orthogonal of $E^{\mathrm{ort}}(M)$ in $E^\times(M)$, i.e. $(E^\times)^{\mathrm{sub}}(M)$, and with kernel the orthogonal of $E^{\mathrm{sub}}(M) = (E^{\times\times})^{\mathrm{sub}}(M)$ in $E^\times(M)$, i.e. $(E^\times)^{\mathrm{ort}}(M)$. Finally, as $P : L_2(M) \to L_2(M)$ is self-adjoint, we easily deduce that $P(x) = P^\times(x)$ for every $x \in E(M) \cap E^\times(M)$. The proof is complete. $\square$

Theorem A (Pisier's method)**.** *Let M be a tracial von Neumann algebra that is equipped with an admissible subspace of $(L_1 + L_\infty)(M)$. We assume that, if N denote the commutative tracial von Neumann algebra introduced in the previous paragraph, there is an admissible subspace of $(L_1 + L_\infty)(M\bar{\otimes}N)$ for which the following two conditions hold.*

1. *the tensor product $L_2^{\mathrm{sub}}(M) \otimes_2 L_2(N)$ coincides with $L_2^{\mathrm{sub}}(M\bar{\otimes}N)$.*
2. *the algebraic tensor product $L_\infty^{\mathrm{ort}}(M) \odot L_\infty(N)$ is included in $L_\infty^{\mathrm{ort}}(M\bar{\otimes}N)$.*

Then, if $1 < p < \infty$ and $0 < \theta < 1$, we have

$$(L_1^{\mathrm{sub}}(M), L_p^{\mathrm{sub}}(M))_\theta = L_{p_\theta}^{\mathrm{sub}}(M)$$

with equivalent norms, with constants depending on p, θ only, where $1/p_\theta = (1 - \theta) + \theta/p$.

If M is a tracial von Neumann algebra equipped with an admissible subspace of $(L_1 + L_\infty)(M)$, and if $1 \le p, q, r \le \infty$ with $r \in [p, q]$, then $L_p^{\mathrm{sub}}(M) \cap L_q^{\mathrm{sub}}(M)$ is clearly a norm-dense subspace of $L_r^{\mathrm{sub}}(M)$ because its contains $(L_1 \cap L_\infty)^{\mathrm{sub}}(M)$. Thus, by the reiteration theorem, from the previous theorem we directly deduce the following corollary.

Corollary 3.16. *Under the setting of the previous theorem, if $1 \le p_0, p_1 < \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}^{\mathrm{sub}}(M), L_{p_1}^{\mathrm{sub}}(M))_\theta = L_{p_\theta}^{\mathrm{sub}}(M)$$

with equivalent norms, with constants depending on p_0, p_1, θ only, where $1/p_\theta = (1 - \theta)/p_0 + \theta/p_1$.

3.3.2 The proof

In this paragraph, we provide a proof of Theorem A.

Let M be a tracial von Neumann algebra with trace τ, and let N be the commutative tracial von Neumann algebra introduced in the previous paragraph. Let $M\bar{\otimes}N$ be the tensor product von Neumann algebra equipped with the tensor product trace τ'. We assume that M and $M\bar{\otimes}N$ are both equipped with a strongly admissible subspace of $(L_1 + L_\infty)(M)$ and $(L_1 + L_\infty)(M\bar{\otimes}N)$ respectively, that satisfy the two conditions of Theorem A. Let $1 < p < \infty$ be fixed. Let $1 < q < \infty$ such that $1/p + 1/q = 1$. For $z \in \overline{\mathbb{B}}$, let $p_z \in [1, p]$ and $q_z \in [q, \infty]$ such that

$$\frac{1}{p_z} = \frac{1 - \mathrm{Re}(z)}{p} + \mathrm{Re}(z), \qquad \frac{1}{q_z} = \frac{1 - \mathrm{Re}(z)}{q}.$$

Note that we have $\frac{1}{p_z}+\frac{1}{q_z}=1$, and if $\mathrm{Re}(z)<1$ then $1<p_z,q_z<\infty$.

If (E_0,E_1) is any compatible couple, we recall the convention $(E_0,E_1)_j:=E_j$ for $j\in\{0,1\}$ and $(E_0,E_1)_{j,r}:=E_j$ for $j\in\{0,1\}$ and $1\leq r\leq\infty$.

Lemma 3.17. *If $z\in\overline{\mathbb{B}}$, we have*

$$L_{p_z}^{\mathrm{sub}}(M)=(L_p^{\mathrm{sub}}(M),L_1^{\mathrm{sub}}(M))_{\mathrm{Re}(z),p_z}$$

with equivalent norms, with constants depending on $p,\mathrm{Re}(z)$ only.

Proof. If $\mathrm{Re}(z)\in\{0,1\}$, there is nothing to prove. If $0<\mathrm{Re}(z)<1$, by the reiteration theorem for the real method, we know that $L_{p_z}(M)=(L_p(M),L_1(M))_{\mathrm{Re}(z),p_z}$ with equivalent norms, with constants depending on $p,\mathrm{Re}(z)$ only, and the result follows as an application of Proposition 3.14. □

The following lemma is proved similarly.

Lemma 3.18. *If $z\in\overline{\mathbb{B}}$, we have*

$$L_{p_z,1}^{\mathrm{sub}}(M\bar{\otimes}N)=(L_{p,1}^{\mathrm{sub}}(M\bar{\otimes}N),L_1^{\mathrm{sub}}(M\bar{\otimes}N))_{\mathrm{Re}(z),1}$$

(with the convention $L_{1,1}^{\mathrm{sub}}(M\bar{\otimes}N):=L_1^{\mathrm{sub}}(M\bar{\otimes}N)$) with equivalent norms, with constants depending on $p,\mathrm{Re}(z)$ only.

The two following lemmas are direct applications of Proposition 3.15.

Lemma 3.19. *If $z\in\overline{\mathbb{B}}$, $\mathrm{Re}(z)<1$, there is a compatible bounded idempotent operator $P:(L_{p_z}+L_{q_z})(M)\to(L_{p_z}+L_{q_z})(M\bar{\otimes}N)$ with constant depending on $p,\mathrm{Re}(z)$ only, such that the bounded idempotent operators $P:L_{p_z}(M)\to L_{p_z}(M)$ and $P:L_{q_z}(M)\to L_{q_z}(M)$ have range $L_{p_z}^{\mathrm{sub}}(M)$ and $L_{q_z}^{\mathrm{sub}}(M)$ respectively, and have kernel $L_{p_z}^{\mathrm{ort}}(M)$ and $L_{q_z}^{\mathrm{ort}}(M)$ respectively. Finally, $P:L_{q_z}(M)\to L_{q_z}(M)$ is the dual of $P:L_{p_z}(M)\to L_{p_z}(M)$ w.r.t. Köthe duality.*

Lemma 3.20. *If $z\in\overline{\mathbb{B}}$, $\mathrm{Re}(z)<1$, there is a compatible bounded idempotent operator $Q:(L_{p_z,1}+L_{q_z,\infty})(M\bar{\otimes}N)\to(L_{p_z,1}+L_{q_z,\infty})(M\bar{\otimes}N)$ with constant depending on $p,\mathrm{Re}(z)$ only, such that the bounded idempotent operators $Q:L_{p_z,1}(M\bar{\otimes}N)\to L_{p_z,1}(M\bar{\otimes}N)$ and $Q:L_{q_z,\infty}(M\bar{\otimes}N)\to L_{q_z,\infty}(M\bar{\otimes}N)$ have range $L_{p_z,1}^{\mathrm{sub}}(M\bar{\otimes}N)$ and $L_{q_z,\infty}^{\mathrm{sub}}(M\bar{\otimes}N)$ respectively, and have kernel $L_{p_z,1}^{\mathrm{ort}}(M\bar{\otimes}N)$ and $L_{q_z,\infty}^{\mathrm{ort}}(M\bar{\otimes}N)$ respectively. Finally, $Q:L_{q_z,\infty}(M\bar{\otimes}N)\to L_{q_z,\infty}(M\bar{\otimes}N)$ is the dual of $Q:L_{p_z,1}(M\bar{\otimes}N)\to L_{p_z,1}(M\bar{\otimes}N)$ w.r.t. Köthe duality.*

As $(L_1 \cap L_\infty)^{\text{sub}}(M)$ is norm-dense in both $L_p^{\text{sub}}(M)$ and $L_1^{\text{sub}}(M)$, the compatible couple $(L_p^{\text{sub}}(M), L_1^{\text{sub}}(M))$ is regular. Thus, we can consider the dual compatible couple $(L_p^{\text{sub}}(M)^*, L_1^{\text{sub}}(M)^*)$ as defined in the preliminary section. Then, there is a compatible contractive operator $I^* : (L_q(M), L_\infty(M)) \to (L_p^{\text{sub}}(M)^*, L_1^{\text{sub}}(M)^*)$ such that

$$I^*(y)(x) = \tau(xy), \quad \text{for } x \in L_p^{\text{sub}}(M) \cap L_1^{\text{sub}}(M),\ y \in (L_q + L_\infty)(M).$$

Moreover, by Hahn-Banach we know that the operators $I^* : L_q(M) \to L_p^{\text{sub}}(M)^*$ and $I^* : L_\infty(M) \to L_1^{\text{sub}}(M)^*$ are both surjective, and as a consequence, if $\mathcal{F}$ is an exact interpolation functor, then we know that I^* induces a contractive surjective operator $\mathcal{F}(L_q(M), L_\infty(M)) \to \mathcal{F}(L_p^{\text{sub}}(M)^*, L_1^{\text{sub}}(M)^*)$.

Lemma 3.21. *If $z \in \overline{\mathbb{B}}$, then I^* induces a bounded surjective operator $L_{q_z}(M) \to L_{p_z}^{\text{sub}}(M)^*$ with kernel $L_{q_z}^{\text{ort}}(M)$, and with a constant depending on $p, \operatorname{Re}(z)$ only.*

Proof. By Lemma 3.17 and the duality theorem for the real method, we know that $L_{p_z}^{\text{sub}}(M)^* = (L_p^{\text{sub}}(M)^*, L_1^{\text{sub}}(M)^*)_{\operatorname{Re}(z),q_z}$ with equivalent norms, with constants depending on $p, \operatorname{Re}(z)$ only. Moreover, by the reiteration theorem for the real method, we have $L_{q_z}(M) = (L_q(M), L_\infty(M))_{\operatorname{Re}(z),q_z}$ with equivalent norms, with constants depending on $p, \operatorname{Re}(z)$ only. As a consequence, I^* induces a bounded surjective operator $L_{q_z}(M) \to L_{p_z}^{\text{sub}}(M)^*$ with constants depending on $p, \operatorname{Re}(z)$. Finally, by definition the kernel of this operator is

$$\Big\{ y \in L_{q_z}(M) \ : \ \tau(xy) = 0,\ \forall x \in L_p^{\text{sub}}(M) \cap L_1^{\text{sub}}(M) \Big\}.$$

As $L_p^{\text{sub}}(M) \cap L_1^{\text{sub}}(M)$ is norm-dense in $L_{p_z}^{\text{sub}}(M)$, we see that the above set coincides with

$$\Big\{ y \in L_{q_z}(M) \ : \ \tau(xy) = 0,\ \forall x \in L_{p_z}^{\text{sub}}(M) \Big\} = L_{q_z}^{\text{ort}}(M).$$

$\square$

As $(L_1 \cap L_\infty)^{\text{sub}}(M \bar{\otimes} N)$ is norm-dense in both $L_{p,1}^{\text{sub}}(M \bar{\otimes} N)$ and $L_1^{\text{sub}}(M \bar{\otimes} N)$, the compatible couple $(L_{p,1}^{\text{sub}}(M \bar{\otimes} N), L_1^{\text{sub}}(M \bar{\otimes} N))$ is regular. Thus, we can consider the dual compatible couple $(L_{p,1}^{\text{sub}}(M \bar{\otimes} N)^*, L_1^{\text{sub}}(M \bar{\otimes} N)^*)$. Then, as before, there is a contractive operator $J^* : (L_{q,\infty}(M \bar{\otimes} N), L_\infty(M \bar{\otimes} N)) \to (L_{p,1}^{\text{sub}}(M \bar{\otimes} N)^*, L_1^{\text{sub}}(M \bar{\otimes} N)^*)$ such that

$$J^*(y')(x') = \tau'(x'y'), \quad \text{for } x' \in L_{p,1}^{\text{sub}}(M \bar{\otimes} N) \cap L_1^{\text{sub}}(M \bar{\otimes} N),\ y' \in (L_{q,\infty} + L_\infty)(M \bar{\otimes} N).$$

The following lemma is proved similarly to the previous one.

Lemma 3.22. *If $z \in \overline{\mathbb{B}}$, then J^* induces a bounded surjective operator $L_{q_z,\infty}(M\bar{\otimes}N) \to L^{\mathrm{sub}}_{p_z,1}(M\bar{\otimes}N)^*$ with kernel $L^{\mathrm{ort}}_{q_z}(M\bar{\otimes}N)$, and with a constant depending on $p, \mathrm{Re}(z)$ only.*

For $z \in \overline{\mathbb{B}}$, let $F_z : L_{q_z}(M) \to L_{q_z,\infty}(M\bar{\otimes}N)$, $x \mapsto x \otimes f_z$ be the amplification operator studied in the previous section. We recall the content of Theorem 3.11.

- for every $x \in (L_q \cap L_\infty)(M)$, the function $\overline{\mathbb{B}} \to L_{q,\infty}(M\bar{\otimes}N) + L_\infty(M\bar{\otimes}N)$, $z \mapsto F_z(x)$ belongs to $\mathcal{F}_*(L_{q,\infty}(M\bar{\otimes}N), L_\infty(M\bar{\otimes}N))$,
- for every $z \in \overline{\mathbb{B}}$ and $x \in L_{q_z}(M)$ we have

$$\|x\|_{L_{q_z}(M)} \leq q_z^{1/q_z} \|F_z(x)\|_{L_{q_z,\infty}(M\bar{\otimes}N)} \leq (1 - q_z^{-1})^{-1} \|x\|_{L_{q_z}(M)}. \tag{3.1}$$

  (with the convention $\infty^0 = 1$).

Proposition 3.23. *If $z \in \overline{\mathbb{B}}$, $\mathrm{Re}(z) < 1$, then we have $Q(F_z(x)) = F_z(P(x))$ for every $x \in L_{q_z}(M)$.*

Proof. By construction, P is compatible with the projection $L_2(M) \to L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ while Q is compatible with the projection $L_2(M\bar{\otimes}N) \to L_2(M\bar{\otimes}N)$ onto $L_2^{\mathrm{sub}}(M\bar{\otimes}N)$. As we have $L_2^{\mathrm{sub}}(M) \otimes_2 L_2(N) = L_2^{\mathrm{sub}}(M\bar{\otimes}N)$ by the hypothesis of Theorem A, we deduce that $Q(x \otimes y) = P(x) \otimes y$ for every $x \in (L_2 \cap L_{q_z})(M)$ and $y \in (L_2 \cap L_{q_z,\infty})(N)$, and thus this extends for every $x \in L_{q_z}(M)$ and $y \in L_{q_z,\infty}(N)$ because $P : L_{q_z}(M) \to L_{q_z}(M)$ and $Q : L_{q_z,\infty}(M\bar{\otimes}N) \to L_{q_z,\infty}(M\bar{\otimes}N)$ are weak*-continuous as they are the dual of $P : L_{p_z}(M) \to L_{p_z}(M)$ and $Q : L_{p_z,1}(M\bar{\otimes}N) \to L_{p_z,1}(M\bar{\otimes}N)$ respectively. In particular, if $x \in L_{q_z}(M)$, we have $Q(F_z(x)) = Q(x \otimes f_z) = P(x) \otimes f_z = F_z(P(x))$, as desired. $\square$

Lemma 3.24. *If $z \in \overline{\mathbb{B}}$, then $F_z : L_{q_z}(M) \to L_{q_z,\infty}(M\bar{\otimes}N)$ maps $L^{\mathrm{ort}}_{q_z}(M)$ into $L^{\mathrm{ort}}_{q_z,\infty}(M\bar{\otimes}N)$.*

Proof. The case $\mathrm{Re}(z) = 1$ holds by the hypothesis of Theorem A. If $\mathrm{Re}(z) < 1$, and if $x \in L^{\mathrm{ort}}_{q_z}(M)$, then $P(x) = 0$, thus $Q(F_z(x)) = F_z(P(x)) = 0$, showing that $F_z(x) \in L^{\mathrm{ort}}_{q_z,\infty}(M\bar{\otimes}N)$. $\square$

If $z \in \overline{\mathbb{B}}$, then as a direct consequence of the above lemma, there is a unique operator $T_z : L^{\mathrm{sub}}_{p_z}(M)^* \to L^{\mathrm{sub}}_{p_z,1}(M\bar{\otimes}N)^*$ that makes the following diagram commute.

$$\begin{array}{ccc} L_{q_z}(M) & \xrightarrow{F_z} & L_{q_z,\infty}(M\bar{\otimes}N) \\ {\scriptstyle I^*}\downarrow & & \downarrow{\scriptstyle J^*} \\ L^{\mathrm{sub}}_{p_z}(M)^* & \xrightarrow{T_z} & L^{\mathrm{sub}}_{p_z,1}(M\bar{\otimes}N)^* \end{array}$$

Lemma 3.25. *If $z \in \overline{\mathbb{B}}$, $\mathrm{Re}(z) < 1$ and $y \in L_{q_z}(M)$, then $I^*(y) = I^*(P(y))$ and*

$$\|I^*(y)\|_{L_{p_z}^{\mathrm{sub}}(M)^*} \geq C\|P(y)\|_{L_{q_z}(M)}$$

where $C > 0$ depends only on $p, \mathrm{Re}(z)$.

Proof. Fix $z \in \overline{\mathbb{B}}$ such that $\mathrm{Re}(z) < 1$ and $y \in L_{q_z}(M)$. Then $P(y) \in L_{q_z}^{\mathrm{sub}}(M)$ and $y - P(y) \in L_{q_z}^{\mathrm{ort}}(M)$. Now, if $x \in L_p^{\mathrm{sub}}(M) \cap L_1^{\mathrm{sub}}(M)$, then $x \in L_{p_z}^{\mathrm{sub}}(M)$, thus $\tau(x(y - P(y)) = 0$, i.e. $\tau(xy) = \tau(xP(y))$, showing that $I^*(y) = I^*(P(y))$. Finally, if $x \in L_p(M) \cap L_1(M)$, then $x - P(x) \in L_{p_z}^{\mathrm{ort}}(M)$, thus $\tau((x - P(x))P(y)) = 0$, i.e. $\tau(xP(y)) = \tau(P(x)P(y))$, and so

$$\begin{aligned}
\|P(y)\|_{L_{q_z}(M)} &= \sup_{\|x\|_{L_{p_z}(M)} \leq 1} |\tau(xP(y))| \\
&= \sup_{\|x\|_{L_{p_z}(M)} \leq 1} |\tau(P(x)P(y))| \\
&= \sup_{\|x\|_{L_{p_z}(M)} \leq 1} |I^*(P(y))(P(x))| \\
&\leq \sup_{\|x\|_{L_{p_z}(M)} \leq 1} \|I^*(P(y))\|_{L_{p_z}^{\mathrm{sub}}(M)^*} \|P(x)\|_{L_{p_z}(M)} \\
&\leq \|I^*(P(y))\|_{L_{p_z}^{\mathrm{sub}}(M)^*} \|P\|_{L_{p_z}(M) \to L_{p_z}(M)}.
\end{aligned}$$

□

The following lemma is proved similarly.

Lemma 3.26. *If $z \in \overline{\mathbb{B}}$, $\mathrm{Re}(z) < 1$ and $y' \in L_{q_z,\infty}(M\bar{\otimes}N)$, then $J^*(y') = J^*(Q(y'))$ and*

$$\|J^*(y')\|_{L_{p_z,1}^{\mathrm{sub}}(M\bar{\otimes}N)^*} \geq C\|Q(y')\|_{L_{q_z,\infty}(M\bar{\otimes}N)}$$

where $C > 0$ depends only on $p, \mathrm{Re}(z)$.

Theorem 3.27. *The family $(T_z)_{z\in\overline{\mathbb{B}}}$ satisfies the following properties.*

1. *if $\phi \in L_p^{\mathrm{sub}}(M)^* \cap L_1^{\mathrm{sub}}(M)^*$, the function $\overline{\mathbb{B}} \to L_{p,1}^{\mathrm{sub}}(M\bar{\otimes}N)^* + L_1^{\mathrm{sub}}(M\bar{\otimes}N)^*$, $z \mapsto T_z(\phi)$ belongs to $\mathcal{F}_*(L_{p,1}^{\mathrm{sub}}(M\bar{\otimes}N)^*, L_1^{\mathrm{sub}}(M\bar{\otimes}N)^*)$,*
2. *for every $z \in \overline{\mathbb{B}}$ and $\phi \in L_{p_z}^{\mathrm{sub}}(M)^*$ we have*

$$\|T_z(\phi)\|_{L_{p_z,1}^{\mathrm{sub}}(M\bar{\otimes}N)^*} \leq C\|\phi\|_{L_{p_z}^{\mathrm{sub}}(M)^*}$$

and if in addition $\mathrm{Re}(z) < 1$, then

$$\|T_z(\phi)\|_{L_{p_z,1}^{\mathrm{sub}}(M\bar{\otimes}N)^*} \geq C'\|\phi\|_{L_{p_z}^{\mathrm{sub}}(M)^*}$$

where $C, C' > 0$ depends only on $p, \mathrm{Re}(z)$.

Proof. *1)* Fix $\phi \in L_p^{\rm sub}(M)^* \cap L_1^{\rm sub}(M)^*$. As I^* induces a surjective operator $I^* : L_q(M) \cap L_\infty(M) \to L_p^{\rm sub}(M)^* \cap L_1^{\rm sub}(M)^*$, there is $y \in (L_q \cap L_\infty)(M)$ such that $\phi = I^*(y)$. If $z \in \overline{\mathbb{B}}$, then by definition of T_z we have $T_z(\phi) = J^*(F_z(y))$. As the operators $J^* : L_{q,\infty}(M\bar{\otimes}N) \to L_{p,1}^{\rm sub}(M\bar{\otimes}N)^*$ and $J^* : L_\infty(M\bar{\otimes}N) \to L_1^{\rm sub}(M\bar{\otimes}N)^*$ are clearly *-weakly continuous as dual operators, and because we know that the function $\overline{\mathbb{B}} \to (L_{q,\infty} + L_\infty)(M\bar{\otimes}N)$, $z \mapsto F_z(y)$ belongs to $\mathcal{F}_*(L_{q,\infty}(M\bar{\otimes}N), L_{q,\infty}(M\bar{\otimes}N))$, by Proposition 3.1 we get the desired conclusion.

2) Fix $z \in \overline{\mathbb{B}}$ and $\phi \in L_{p_z}^{\rm sub}(M)^*$. Then, by Hahn-Banach there is $y \in L_{q_z}(M)$ such that $\phi = I^*(y)$ and $\|y\|_{L_{q_z}(M)} = \|\phi\|_{L_{p_z}^{\rm sub}(M)^*}$. As $T_z(\phi) = J^*(F_z(y))$, by (3.1) we get

$$\begin{aligned}\|T_z(\phi)\|_{L_{p_z,1}^{\rm sub}(M\bar{\otimes}N)^*} &\leq \|J^*\|_{L_{q_z,\infty}(M\bar{\otimes}N)\to L_{p_z,1}^{\rm sub}(M\bar{\otimes}N)^*}\|(F_z(y))\|_{L_{q_z,\infty}(M\bar{\otimes}N)}\\ &\leq \|J^*\|_{L_{q_z,\infty}(M\bar{\otimes}N)\to L_{p_z,1}^{\rm sub}(M\bar{\otimes}N)^*} q_z^{-1/q_z}(1-q_z^{-1})^{-1}\|y\|_{L_{q_z}(M)}.\end{aligned}$$

if in addition $\mathrm{Re}(z) < 1$, then by Lemma 3.26 we have

$$T_z(\phi) = J^*(F_z(y)) = J^*(Q(F_z(y)) = J^*(F_z(P(y)))$$

and thus, by again Lemma 3.26 and (3.1) we get

$$\|T_z(\phi)\|_{L_{p_z,1}^{\rm sub}(M)^*} \geq C\|F_z(P(y))\|_{L_{q_z,\infty}(M)} \geq Cq_z^{-1/q_z}\|P(y)\|_{L_{q_z}(M)}$$

where $C > 0$ depends only on $p, \mathrm{Re}(z)$. Moreover, by Lemma 3.26 we have $\phi = I^*(y) = I^*(P(y))$, and thus

$$\|\phi\|_{L_{p_z}^{\rm sub}(M)^*} \leq \|I^*\|_{L_{q_z}(M)\to L_{p_z}^{\rm sub}(M)^*}\|P(y)\|_{L_{q_z}(M)}.$$

The proof is complete. □

Finally, we are now able to complete the proof of Theorem A.

Proof of Theorem A. Fix $\phi \in L_p^{\rm sub}(M)^* \cap L_1^{\rm sub}(M)^*$ and $0 < \theta < 1$. By the first two points of Theorem 3.27, we can apply the weak*-Stein theorem with the family $(T_z)_{z\in\overline{\mathbb{B}}}$ to deduce that $T_\theta(\phi) \in (L_{p,1}^{\rm sub}(M\bar{\otimes}N)^*, L_1^{\rm sub}(M\bar{\otimes}N)^*)_{\theta,\infty}$ with

$$\|T_\theta(x)\|_{(L_{p,1}^{\rm sub}(M\bar{\otimes}N)^*, L_1^{\rm sub}(M\bar{\otimes}N)^*)_{\theta,\infty}} \leq C\|x\|_{(L_p^{\rm sub}(M)^*, L_1^{\rm sub}(M)^*)_\theta}$$

where $C > 0$ depends on p, θ only. By Lemma 3.18, we deduce that

$$\|T_\theta(x)\|_{L_{p_\theta,1}^{\rm sub}(M\bar{\otimes}N)^*} \leq C'\|x\|_{(L_p^{\rm sub}(M)^*, L_1^{\rm sub}(M)^*)_\theta}$$

where $C' > 0$ depends on p, θ only. Finally, by the last point of Theorem 3.27, we deduce that

$$\|x\|_{L_{p_\theta}^{\mathrm{sub}}(M)^*} \leq C''\|x\|_{(L_p^{\mathrm{sub}}(M)^*, L_1^{\mathrm{sub}}(M)^*)_\theta}$$

where $C'' > 0$ depends on p, θ only. As $L_p^{\mathrm{sub}}(M)^* \cap L_1^{\mathrm{sub}}(M)^*$ is norm-dense in $(L_p^{\mathrm{sub}}(M)^*, L_1^{\mathrm{sub}}(M)^*)_\theta$, this shows that we have a continuous inclusion

$$(L_p^{\mathrm{sub}}(M)^*, L_1^{\mathrm{sub}}(M)^*)_\theta \subset L_{p_\theta}^{\mathrm{sub}}(M)^*$$

with a constant depending on p, θ only. By Corollary 2.12, we deduce that we have a continuous inclusion

$$L_{p_\theta}^{\mathrm{sub}}(M) \subset (L_p^{\mathrm{sub}}(M), L_1^{\mathrm{sub}}(M))_\theta$$

with a constant depending on p, θ only. As we clearly have a contractive inclusion in the converse way, the proof is complete. $\square$

4 Main results

4.1 Theorem B

Let M be a tracial von Neumann algebra equipped with a filtration $(M_n)_{n\geq 1}$ with associated conditional expectations denoted $(E_n)_{n\geq 1}$ and associated increment projections denoted $(D_n)_{n\geq 1}$. Let I be a fixed set of positive integer. We set

$$(L_1 + L_\infty)^{\mathrm{sub}}(M) := \Big\{ x \in (L_1 + L_\infty)(M) \ : \ \forall n \notin I,\ D_n(x) = 0 \Big\}.$$

In accordance with the notations of the previous section, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, we set

$$E^{\mathrm{sub}}(M) := E(M) \cap (L_1 + L_\infty)^{\mathrm{sub}}(M) = \Big\{ x \in E(M) \ : \ \forall n \notin I,\ D_n(x) = 0 \Big\}.$$

It is clear that $E^{\mathrm{sub}}(M)$ is a weakly closed subspace of $E(M)$ w.r.t. Köthe duality, and in addition it is stabilised by E_n for every $n \geq 1$.

Proposition 4.1. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then*

$$\Big\{ x \in \cup_{n\geq 1} (L_1 \cap L_\infty)(M_n) \ : \ \forall n \notin I,\ D_n(x) = 0 \Big\}$$

is a norm-dense subspace $E^{\mathrm{sub}}(M)$ and a weak-dense subspace of $(E^\times)^{\mathrm{sub}}(M)$.*

Proof. Fix $x \in E^{\mathrm{sub}}(M)$. Then we know that the sequence $(E_n(x))_{n\geq 1}$ belongs to $E^{\mathrm{sub}}(M)$, and by By Theorem 2.23 it converges in norm to x in $E(M)$. Thus we can assume that there is $n \geq 1$ such that $E_n(x) = x$, so that we have

$$x = \sum_{k=1}^{n} D_k(x) = \sum_{k \in I, k \leq n} D_k(x).$$

As $(L_1 \cap L_\infty)(M)$ is norm dense in $E(M)$, there is a net $(y_\alpha)_\alpha$ of $(L_1 \cap L_\infty)(M)$ that converges in norm to x in $E(M)$ (resp. *weakly to x in $E^\times(M)$). We set

$$x_\alpha := \sum_{k \in I, k \leq n} D_k(y_\alpha).$$

Then $x_\alpha \in (L_1 \cap L_\infty)(M_n)$ and $D_n(x_\alpha) = 0$ for every $n \notin I$. As the net $(x_\alpha)_\alpha$ converges in norm to x in $E(M)$, the proof is complete. The statement for $(E^\times)^{\mathrm{sub}}(M)$ is derived analogously. $\square$

In accordance with the notations of the previous section, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, we denote $E^{\text{ort}}(M)$ the orthogonal of $(E^\times)^{\text{sub}}(M)$ in $E(M)$ w.r.t. Köthe duality.

Proposition 4.2. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then*

$$E^{\text{ort}}(M) = \Big\{x \in E(M) \ : \ \forall n \in I,\ D_n(x) = 0\Big\}.$$

and

$$(E^\times)^{\text{ort}}(M) = \Big\{x \in (E^\times)(M) \ : \ \forall n \in I,\ D_n(x) = 0\Big\}.$$

Proof. If $x \in E(M)$ is such that $D_n(x) = 0$ for every $n \in I$, then for $y \in (E^\times)^{\text{sub}}(M)$ we have

$$\tau(xy) = \sum_{n\geq 1} \tau(D_n(x)D_n(y)) = 0.$$

In the converse way, if $x \in E^{\text{ort}}(M)$, and if $n \in I$, then for every $y \in E^\times(M)$ we clearly have $D_n(y) \in (E^\times)^{\text{sub}}(M)$ so that

$$\tau(D_n(x)y) = \tau(xD_n(y)) = 0$$

and as a consequence $D_n(x) = 0$, as desired. The expression for $(E^\times)^{\text{sub}}(M)$ is derived analogously. □

The proof of the following proposition is straightforward.

Proposition 4.3. *Let $P : L_2(M) \to L_2(M)$ denote the orthogonal projection onto $L_2^{\text{sub}}(M)$. Then for every $x \in L_2(M)$, we have*

$$P(x) = \sum_{n\in I} D_n(x), \quad \textit{in } L_2(M).$$

Theorem B. *If $1 \leq p_0, p_1 < \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}^{\text{sub}}(M), L_{p_1}^{\text{sub}}(M))_\theta = L_{p_\theta}^{\text{sub}}(M)$$

with equivalent norms, with constants depending on p_0, p_1, θ only, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

The proof of Theorem B is based on Pisier's method.

Lemma 4.4. *$(L_1 + L_\infty)^{\mathrm{sub}}(M)$ is an admissible subspace of $(L_1 + L_\infty)^{\mathrm{sub}}(M)$.*

Proof. • If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then we know that $E^{\mathrm{sub}}(M)$ is weakly closed in $E(M)$ w.r.t. Köthe duality and in particular it is norm-closed in $E(M)$.

• If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm, then Proposition 4.1 shows that $(L_1 \cap L_\infty)^{\mathrm{sub}}(M)$ is norm-dense in $E(M)$ and weak*-dense in $E^\times(M)$.

• If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then by Proposition 4.2 and by Proposition 4.1 applied with the complement subset of I instead of I shows that $(L_1 \cap L_\infty)(M) \cap E^{\mathrm{ort}}(M)$ is norm-dense in $E^{\mathrm{ort}}(M)$.

• By Proposition 4.3, we see that the orthogonal projection $P : L_2(M) \to L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ is a martingale transform. Thus, as a consequence of the boundedness properties of martingale transforms as proved in [9], we deduce that, if $1 < p < \infty$, then $P : L_2(M) \to L_2(M)$ is L_p-bounded with a constant depending on p only.

• Finally, in [6][Theorem 2.8] it is proved that, if $1 \le p, q \le \infty$, then the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in the compatible couple $(L_p(M), L_q(M))$ with a universal constant. $\square$

Let N be the commutative tracial von Neumann algebra introduced in the previous paragraph. Let $M \bar{\otimes} N$ be the tensor product von Neumann algebra equipped with the tensor product trace, and set

$$(L_1 + L_\infty)^{\mathrm{sub}}(M \bar{\otimes} N) := \Big\{ x \in (L_1 + L_\infty)(M) \ : \ \forall n \notin I, \ (D_n \otimes I)(x) = 0 \Big\}.$$

Note that $(D_n \otimes I)_{n \ge 1}$ correspond to the increment projections associated with the filtration $(M_n \bar{\otimes} N)_{n \ge 1}$ on $M \bar{\otimes} N$. In particular, the above arguments apply again, so that we immediately have the following result.

Lemma 4.5. *$(L_1 + L_\infty)^{\mathrm{sub}}(M \bar{\otimes} N)$ is an admissible subspace of $(L_1 + L_\infty)^{\mathrm{sub}}(M \bar{\otimes} N)$.*

Hence, it only remains to check the hypothesis of Theorem A.

Lemma 4.6. *The following two assertions are valid.*

1. *the tensor product $L_2^{\mathrm{sub}}(M) \otimes_2 L_2(N)$ coincides with $L_2^{\mathrm{sub}}(M \bar{\otimes} N)$.*
2. *the algebraic tensor product $L_\infty^{\mathrm{ort}}(M) \odot L_\infty(N)$ is included in $L_\infty^{\mathrm{ort}}(M \bar{\otimes} N)$.*

Proof. By definition, we have

$$L_2^{\mathrm{sub}}(M) = \Big\{ x \in L_2(M) \ : \ \forall n \notin I, \ D_n(x) = 0 \Big\},$$

$$L_2^{\text{sub}}(M\bar{\otimes}N) = \Big\{x \in L_2(M\bar{\otimes}N) \ : \ \forall n \notin I,\ (D_n \otimes I)(x) = 0\Big\},$$

and by Proposition 4.2, we have

$$L_\infty^{\text{ort}}(M) = \Big\{x \in L_\infty(M) \ : \ \forall n \in I,\ D_n(x) = 0\Big\},$$

$$L_\infty^{\text{ort}}(M\bar{\otimes}N) = \Big\{x \in L_\infty(M\bar{\otimes}N) \ : \ \forall n \in I,\ (D_n \otimes I)(x) = 0\Big\}.$$

The lemma follows directly from the four above expressions. □

4.2 Theorem C

Let M be a tracial von Neumann algebra equipped with two filtrations $(M_n^-)_{n\geq 1}$, $(M_n^+)_{n\geq 1}$ with associated conditional expectations denoted by $(E_n^-)_{n\geq 1}$, $(E_n^+)_{n\geq 1}$ and increment projections denoted by $(D_n^-)_{n\geq 1}$, $(D_n^+)_{n\geq 1}$ respectively. Let I^-, I^+ be two fixed sets of positive integers. We add the two following assumptions.

Commutation Assumption. The filtrations $(M_n^-)_{n\geq 1}$, $(M_n^+)_{n\geq 1}$ *commute* in the sense that for every $m, n \geq 1$, we have

$$E_m^- E_n^+ = E_n^+ E_m^-. \tag{4.1}$$

Orthogonality Assumption. For every $m \notin I^-$ and $n \notin I^+$, we have

$$D_m^- D_n^+ = D_n^+ D_m^- = 0. \tag{4.2}$$

We set

$$(L_1 + L_\infty)^{\text{sub}}(M) :=$$
$$\Big\{x \in (L_1 + L_\infty)(M) \ : \ \forall n \notin I^-,\ D_n^-(x) = 0,\ \forall n \notin I^+,\ D_n^+(x) = 0\Big\}.$$

In accordance with the notations of the previous section, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ we set

$$E^{\text{sub}}(M) := E(M) \cap (L_1 + L_\infty)^{\text{sub}}(O)$$

$$= \Big\{x \in E(M) \ : \ \forall n \notin I^-,\ D_n^-(x) = 0,\ \forall n \notin I^+,\ D_n^+(x) = 0\Big\}.$$

Again, it is clear that $E^{\text{sub}}(M)$ is a weakly closed subspace of $E(M)$ w.r.t. Köthe duality, and in addition it is stabilised by $E_n^\pm$ for every $n \geq 1$.

Lemma 4.7. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm.*

1. *If $x \in E(M)$ then $(E_n^- E_n^+(x))_{n\geq 1}$ converges in norm to x in $E(M)$ w.r.t. Köthe duality.*
2. *If $y \in E^\times(M)$ then $(E_n^- E_n^+(y))_{n\geq 1}$ converges *-weakly to y in $E^\times(M)$ w.r.t. Köthe duality.*

Proof. If $x \in E(M)$, then

$$\begin{aligned}\|E_n^- E_n^+(x) - x\|_{E(M)} &= \|E_n^-(E_n^+(x) - x) + E_n^-(x) - x\|_{E(M)} \\ &\leq \|E_n^+(x) - x\|_{E(M)} + \|E_n^-(x) - x\|_{E(M)} \underset{n\to\infty}{\to} 0.\end{aligned}$$

Now, if $y \in E^\times(M)$ and $x \in E(M)$ then

$$\tau(xE_n^- E_n^+(y)) = \tau(E_n^- E_n^+(x)y) \underset{n\to\infty}{\to} \tau(xy).$$

□

Proposition 4.8. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then*

$$\Big\{x \in \cup_{n\geq 1}(L_1 \cap L_\infty)(M_n^- \cap M_n^+) \ : \ \forall n \notin I^-,\ D_n^-(x) = 0,\ \forall n \notin I^+,\ D_n^+(x) = 0\Big\}$$

is a norm-dense subspace of $E^{\mathrm{sub}}(M)$ and a weak-dense subspace of $(E^\times)^{\mathrm{sub}}(M)$.*

Proof. Fix $x \in E^{\mathrm{sub}}(M)$. Then we know that the sequence $(E_n^- E_n^+(x))_{n\geq 1}$ belongs to $E^{\mathrm{sub}}(M)$, and by the previous lemma it converges in norm to x in $E(M)$. Thus, by the Commutative Assumption we can assume that there is $n \geq 1$ such that $E_n^-(x) = E_n^+(x) = x$, so that we have

$$x = \sum_{i,j=1}^{n} D_i^- D_j^+(x) = \sum_{i\in I^-, j\in I^+, i,j\leq n} D_i^- D_j^+(x).$$

As $(L_1 \cap L_\infty)(M)$ is norm-dense in $E(M)$, there is a net $(y_\alpha)_\alpha$ of $(L_1 \cap L_\infty)(M)$ that converges in norm to x in $E(M)$. We set

$$x_\alpha := \sum_{i\in I^-, j\in I^+, i,j\leq n} D_i^- D_j^+(y_\alpha).$$

Then $x_\alpha \in (L_1 \cap L_\infty)(M_n)$ with $D_n^-(x) = 0$ for all $n \notin I^-$ and $D_n^+(x) = 0$ for all $n \notin I^+$. As the net $(x_\alpha)_\alpha$ converges in norm to x in $E(M)$, the proof is complete. The statement for $(E^\times)^{\mathrm{sub}}(M)$ is derived analogously. □

In accordance with the notations of the previous section, if $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, we denote $E^{\mathrm{ort}}(M)$ the orthogonal of $(E^\times)^{\mathrm{sub}}(M)$ in $E(M)$ w.r.t. Köthe duality.

Proposition 4.9. *Let $E(M)$ be an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm. Then $(L_1 \cap L_\infty)(M) \cap E^{ort}(M)$ is norm dense in $E^{ort}(M)$.*

Proof. We set

$$E_\pm^{\mathrm{sub}}(M) := \Big\{x \in E(M) \ : \ \forall n \notin I^\pm,\ D_n^\pm(x) = 0\Big\}.$$

$$E_\pm^{\mathrm{ort}}(M) := \Big\{x \in E(M) \ : \ \forall n \in I^\pm,\ D_n^\pm(x) = 0\Big\}.$$

As we know that $E_\pm^{\mathrm{ort}}(M)$ is the orthogonal of $(E^\times)_\pm^{\mathrm{sub}}(M)$ in $E(M)$, and because by definition we have $(E^\times)^{\mathrm{sub}}(M) = (E^\times)_-^{\mathrm{sub}}(M) \cap (E^\times)_+^{\mathrm{sub}}(M)$, we directly deduce that $E^{\mathrm{ort}}(M)$ is the norm closure of $E_-^{\mathrm{ort}}(M) + E_+^{\mathrm{ort}}(M)$ in $E(M)$. Besides, in the previous paragraph we proved that $(L_1)_\pm^{\mathrm{ort}}(M) \cap (L_\infty)_\pm^{\mathrm{ort}}(M)$ is a norm-dense subspace of $E_\pm^{\mathrm{ort}}(M)$ in $E(M)$. Thus, we find that

$$(L_1)_-^{\mathrm{ort}}(M) \cap (L_\infty)_-^{\mathrm{ort}}(M) + (L_1)_+^{\mathrm{ort}}(M) \cap (L_\infty)_+^{\mathrm{ort}}(M)$$

is a norm-dense subspace of $E^{\mathrm{ort}}(M)$. Hence, to conclude, it suffices to check that $(L_1 \cap L_\infty)(M) \cap E^{\mathrm{ort}}(M)$ contains

$$(L_1)_-^{\mathrm{ort}}(M) \cap (L_\infty)_-^{\mathrm{ort}}(M) + (L_1)_+^{\mathrm{ort}}(M) \cap (L_\infty)_+^{\mathrm{ort}}(M).$$

But it is clear that $(L_1 \cap L_\infty)(M) \cap E^{\mathrm{ort}}(M)$ contains

$$\Big[(L_1)_-^{\mathrm{ort}}(M) + (L_1)_+^{\mathrm{ort}}(M)\Big] \cap \Big[(L_\infty)_-^{\mathrm{ort}}(M) + (L_\infty)_+^{\mathrm{ort}}(M)\Big]$$

and the latter clearly contains

$$(L_1)_-^{\mathrm{ort}}(M) \cap (L_\infty)_-^{\mathrm{ort}}(M) + (L_1)_+^{\mathrm{ort}}(M) \cap (L_\infty)_+^{\mathrm{ort}}(M).$$

The proof is complete. □

Proposition 4.10. *Let P denote the orthogonal projection on $L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$. Let $P_\pm$ denote the orthogonal projection on $L_2(M)$ onto $(L_2)_\pm^{\mathrm{sub}}(M)$. Then $P = P_-P_+ = P_+P_-$. Moreover, if $x \in L_2(M)$, then*

$$P_\pm(x) = \sum_{n \in I^\pm} D_n^\pm(x), \quad \text{in } L_2(M).$$

From now on, we assume that $I^- = \{2n-1 \ : \ n \geq 1\}$, $I^+ = \{2n \ : \ n \geq 1\}$, and that there are tracial von Neumann algebras A and B equipped with filtrations $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ such that we have $M = A\bar{\otimes}B$, and such that, for $n \geq 1$ we have

$$\begin{aligned} M_{2n-1}^- &:= A_n\bar{\otimes}B_n, & M_{2n}^- &:= A_{n+1}\bar{\otimes}B_n, \\ M_{2n-1}^+ &:= A_n\bar{\otimes}B_{n+1}, & M_{2n}^+ &:= A_{n+1}\bar{\otimes}B_{n+1}. \end{aligned} \tag{4.3}$$

Then, under the assumption, we have the following result.

Theorem C. *If* $1 \leq p_0, p_1 < \infty$ *and* $0 < \theta < 1$ *then*

$$(L_{p_0}^{\mathrm{sub}}(M), L_{p_1}^{\mathrm{sub}}(M))_\theta = L_{p_\theta}^{\mathrm{sub}}(M)$$

with equivalent norms, with constants depending on p_0, p_1, θ *only, where* $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Again, the proof of Theorem C is based on Pisier's method.

Lemma 4.11. $(L_1 + L_\infty)^{\mathrm{sub}}(M)$ *is an admissible subspace of* $(L_1 + L_\infty)^{\mathrm{sub}}(M)$.

Proof. • If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then we know that $E^{\mathrm{sub}}(M)$ is weakly closed in $E(M)$ w.r.t. Köthe duality and in particular it is norm-closed in $E(M)$.

• If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$ with order continuous norm, then Proposition 4.8 shows that $(L_1 \cap L_\infty)^{\mathrm{sub}}(M)$ is norm-dense in $E(M)$ and weak*-dense in $E^\times(M)$.

• If $E(M)$ is an exact interpolation space for $(L_1(M), L_\infty(M))$, then by Proposition 4.9 we know that $(L_1 \cap L_\infty)(M) \cap E^{\mathrm{ort}}(M)$ is norm-dense in $E^{\mathrm{ort}}(M)$.

• By Proposition 4.10, we see that the orthogonal projection $P : L_2(M) \to L_2(M)$ onto $L_2^{\mathrm{sub}}(M)$ is the composition of two martingale transforms. Thus, as a consequence of the boundedness properties of martingale transforms as proved in [9], we deduce that, if $1 < p < \infty$, then $P : L_2(M) \to L_2(M)$ is L_p-bounded with a constant depending on p only.

• Finally, in [6][Theorem 2.15] it is proved that, if $1 \leq p, q \leq \infty$, then the subcouple $(L_p^{\mathrm{sub}}(M), L_q^{\mathrm{sub}}(M))$ is K-closed in the compatible couple $(L_p(M), L_q(M))$ with a universal constant. □

Let N be the commutative tracial von Neumann algebra introduced in the previous paragraph. Let $M\bar{\otimes}N$ be the tensor product von Neumann algebra equipped with the tensor product trace, and set

$$(L_1 + L_\infty)^{\mathrm{sub}}(M\bar{\otimes}N) :=$$

$$\Big\{x \in (L_1 + L_\infty)(M\bar{\otimes}N) \ : \ \forall n \notin I^-, \ (D_n^- \otimes I)(x) = 0, \ \forall n \notin I^+, \ (D_n^+ \otimes I)(x) = 0\Big\}.$$

Note that $(D_n^\pm \otimes I)_{n\geq 1}$ correspond to the increment projections associated with the filtrations $(M_n^\pm \bar{\otimes} N)_{n\geq 1}$ on $M\bar{\otimes}N$, and these two filtrations satisfy the same commutations and orthogonality assumptions as satisfied by the two filtrations $(M_n^\pm)_{n\geq 1}$. As a consequence, we immediately have the following result.

Lemma 4.12. *$(L_1+L_\infty)^{\mathrm{sub}}(M\bar{\otimes}N)$ is an admissible subspace of $(L_1+L_\infty)^{\mathrm{sub}}(M\bar{\otimes}N)$.*

Hence, it only remains to check the hypothesis of Theorem A.

Lemma 4.13. *The following two assertions are valid.*

1. *the tensor product $L_2^{\mathrm{sub}}(M)\otimes_2 L_2(N)$ coincides with $L_2^{\mathrm{sub}}(M\bar{\otimes}N)$.*
2. *the algebraic tensor product $L_\infty^{\mathrm{ort}}(M)\odot L_\infty(N)$ is included in $L_\infty^{\mathrm{ort}}(M\bar{\otimes}N)$.*

Proof. The first assertion is obvious. Now, fix $x\in L_\infty^{\mathrm{ort}}(M)$ and $f\in L_\infty(N)$. By Proposition 4.8, it suffices to check that $x\otimes f\in L_\infty(M\bar{\otimes}N)$ is orthogonal to any $w\in L_1^{\mathrm{sub}}(M\bar{\otimes}N)$ such that $w\in L_1((M_n^-\otimes N)\cap(M_n^+\otimes N))$ for a certain $n\geq 1$. Thus, consider $n\geq 1$ and $w\in L_1^{\mathrm{sub}}(M\bar{\otimes}N)$ such that $w\in L_1((M_n^-\otimes N)\cap(M_n^+\otimes N))$. Then we have $w=(E_n^-\otimes I)(w)=(E_n^+\otimes I)(w)$, so that we can write

$$w=\sum_{i,j=1}^{n}(D_i^-\otimes I)(D_j^+\otimes I)(w)=\sum_{i\in I^-,j\in I^+,i,j\leq n}(D_i\otimes I)^-(D_j^+\otimes I)(w).$$

Besides, as the algebraic tensor product $L_1(M)\odot L_1(N)$ is norm-dense in $L_1(M\bar{\otimes}N)$, there is a net $(w_\alpha)_\alpha$ of $L_1(M)\odot L_1(N)$ that converges in norm to w in $L_1(M\bar{\otimes}N)$. Finally, we set

$$w'_\alpha:=\sum_{i\in I^-,j\in I^+,i,j\leq n}(D_i\otimes I)^-(D_j^+\otimes I)(w_\alpha).$$

Then clearly the net $(w'_\alpha)_\alpha$ also converges to w in $L_1(M\bar{\otimes}N)$. Thus, it suffices to check that $(\tau\otimes\mu)((x\otimes f)w'_\alpha)=0$. But we can write

$$w_\alpha=\sum_\beta y_{\alpha\beta}\otimes g_{\alpha\beta}$$

with $y_{\alpha\beta}\in L_1(M)$ and $g_{\alpha\beta}\in L_1(N)$. Thus, we have

$$\begin{aligned}
w'_\alpha&=\sum_\beta\sum_{i\in I^-,j\in I^+,i,j\leq n}(D_i\otimes I)^-(D_j^+\otimes I)(y_{\alpha\beta}\otimes g_{\alpha\beta})\\
&=\sum_\beta\sum_{i\in I^-,j\in I^+,i,j\leq n}D_i^-D_j^+(y_{\alpha\beta})\otimes g_{\alpha\beta}\\
&=\sum_\beta\Big[\sum_{i\in I^-,j\in I^+,i,j\leq n}D_i^-D_j^+(y_{\alpha\beta})\Big]\otimes g_{\alpha\beta}\\
&=\sum_\beta y'_{\alpha\beta}\otimes g_{\alpha\beta}
\end{aligned}$$

where

$$y'_{\alpha\beta}:=\sum_{i\in I^-,j\in I^+,i,j\leq n}D_i^-D_j^+(y_{\alpha\beta})$$

clearly belongs to $L_1^{\mathrm{sub}}(M)$. Thus, we have $\tau(xy'_{\alpha\beta}) = 0$ and then

$$\begin{aligned}(\tau \otimes \mu)((x \otimes f)w'_\alpha) &= \sum_\beta (\tau \otimes \mu)((x \otimes f)(y'_{\alpha\beta} \otimes g_{\alpha\beta})) \\ &= \sum_\beta (\tau \otimes \mu)(xy'_{\alpha\beta} \otimes fg_{\alpha\beta}) \\ &= \sum_\beta \tau(xy'_{\alpha\beta})\mu(fg_{\alpha\beta}) = 0\end{aligned}$$

as desired. $\square$

5 Applications

Let M be a tracial von Neumann algebra equipped with a filtration $(M_n)_{n\geq 1}$. For $1 \leq p \leq \infty$, let $L_p(M, \ell_2)$ denote the row/column/mixed space as defined in [6].

Recall that a sequence $(x_n)_{n\geq 1}$ of $(L_1 + L_\infty)(M)$ is said to be adapted if $E_n(x_n) = x_n$ for every $n \geq 1$, and is said to be a martingale increment if it is adapted and if $E_{n-1}(x_n) = 0$ for every $n \geq 2$.

For $1 \leq p < \infty$, we set

$$L_p^{\mathrm{mi}}(M, \ell_2) := \Big\{ x \in L_p(M, \ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment with } x_1 = 0 \Big\}.$$

It is clear that they are closed subspaces of $L_p(M, \ell_2)$. Using the transference techniques used in [6], the following result is are direct consequence of Theorem C.

Theorem 5.1. *If $1 \leq p_0, p_1 < \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}^{\mathrm{mi}}(M), L_{p_1}^{\mathrm{mi}}(M))_\theta = L_{p_\theta}^{\mathrm{mi}}(M)$$

with equivalent norms, with constants depending on p_0, p_1, θ only, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

For $1 \leq p < \infty$, we set

$$L_p^{\mathrm{hardy}}(M, \ell_2) := \Big\{ x \in L_p(M, \ell_2) \ \mid \ (x_n)_{n\geq 1} \ \text{ is a martingale increment} \Big\}.$$

For every $x_1 \in (L_1 + L_\infty)(M)$, let $B(x_1)$ denote the element of $L_1(M, \ell_2) + L_\infty(M, \ell_2)$ such that $B(x_1) = (x_1, 0, 0, \ldots)$.

Corollary 5.2. *If $1 \leq p_0, p_1 < \infty$ and $0 < \theta < 1$ then*

$$(L_{p_0}^{\mathrm{hardy}}(M, \ell_2), L_{p_1}^{\mathrm{hardy}}(M, \ell_2))_\theta = L_{p_\theta}^{\mathrm{hardy}}(M, \ell_2)$$

with equivalent norms, with constants depending on p_0, p_1, θ only, where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$.

Proof. Fix $x \in L_{p_\theta}^{\mathrm{hardy}}(M, \ell_2)$ and $\epsilon > 0$. Then clearly $x - B(x_1) \in L_{p_\theta}^{\mathrm{mi}}(M, \ell_2)$. Thus, by the previous theorem there is $f \in \mathcal{F}(L_{p_0}^{\mathrm{mi}}(M, \ell_2), L_{p_1}^{\mathrm{mi}}(M, \ell_2))$ such that $f(\theta) = x - B(x_1)$ and

$$\max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} \|f(j + is)\|_{L_{p_j}(M,\ell_2)} \leq C\|x - B(x_1)\|_{L_{p_\theta}(M,\ell_2)} + \epsilon \leq C\|x\|_{L_{p_\theta}(M,\ell_2)} + \epsilon$$

where $C > 0$ is a constant depending on p_0, p_1, θ only. Moreover, as we know that $L_{p_\theta}(M) = (L_{p_0}(M), L_{p_1}(M))_\theta$ with equal norms, there is $g \in \mathcal{F}(L_{p_0}(M), L_{p_1}(M))$ such that $g(\theta) = x_1$ and

$$\max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} \|g(j+is)\|_{L_{p_j}(M)} \leq \|x_1\|_{L_{p_\theta}(M)} + \epsilon.$$

Finally, let $h \in \mathcal{F}(L_{p_0}^{\mathrm{hardy}}(M,\ell_2), L_{p_1}^{\mathrm{hardy}}(M,\ell_2))$ such that

$$h(z) = f(z) + B(E_1(g(z))), \qquad z \in \overline{\mathbb{B}}.$$

Then we have

$$h(\theta) = f(\theta) + B(E_1(g(\theta))) = x - B(x_1) + B(E(x_1)) = x$$

and clearly

$$\begin{aligned}\max_{j\in\{0,1\}} \sup_{s\in\mathbb{R}} \|h(j+is)\|_{L_{p_j}^{\mathrm{sub}}(M)} &\leq (C\|x\|_{L_{p_\theta}(M,\ell_2)} + \epsilon) + (\|x_1\|_{L_{p_\theta}(M)} + \epsilon) \\ &\leq (C+1)\|x\|_{L_{p_\theta}(M,\ell_2)} + 2\epsilon.\end{aligned}$$

The conclusion is straightforward. $\square$

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